The maximum number of cycles in a graph of large rank

The formulas of Horak and Kim are exact for every large rank

KNT · Astra · Fable 5.2

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Let $\psi(r)$ be the largest number of cycles in a graph of cyclomatic number r. We prove that, for all sufficiently large r,

$$\psi(2q+1) = 2^{2q} + \left(2q + \tfrac{1}{2}\right)2^q - 3q, \qquad \psi(2q+2) = 2^{2q+1} + \left(2q + \tfrac{9}{2}\right)2^q - 3q - 4.$$

Cubic graphs of Horak and Kim have these numbers of cycles. In particular $\psi(r) \sim 2^{r-1}$, which Entringer and Slater conjectured in 1981. The proof uses the forest estimate of the companion paper [4]. Astra and Fable 5.2 are language models. Section 1.4 states the part of each author. The theorem has a formal proof in Lean 4, and the Lean kernel has checked it.

The proof in seven sentences

1. Subdividing an edge keeps the number of cycles, and splitting a vertex of degree four or more does not lower it. So some cubic multigraph G of rank r has $\psi(r)$ cycles.
2. Fix a small connected set P of vertices. For a cycle that meets P, the edges without an end in P form a forest. By the *forest estimate* of [4], at most $\frac{1}{2} + o(1)$ of all even subgraphs have this property, and few cycles avoid P. So $\psi(r) \le (\frac{1}{2} + o(1))2^r$.
3. A cut with at most three edges, or a cycle of length four, divides G into two parts, and a cycle that meets both parts consists of paths in the parts. A part with more than half of the possible paths between two ends, by a fixed margin, is a ladder up to a bounded error, and it has few cycles.
4. So a graph with such a cut or such a cycle is a ring of copies of $K_{2,2}$ and of single edges, or it has at most $2^{r-1} + O(2^{r/2})$ cycles.
5. A ring has at most $H(r)$ cycles, by a direct count; $H(r)$ is the right side of the formulas above.
6. In the other case G has girth at least five, and it has fewer than 2^{r-1} cycles: by a second-moment argument, G contains two large pieces with small boundaries, and each piece has a forest proportion of at most $\frac{3}{5}$.
7. $H(r) - 2^{r-1}$ is at least $(r-2)2^{r/2-1}$. So $\psi(r) = H(r)$ for large r.

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1 The result and the plan

1.1 The result

Throughout, $\log$ is the binary logarithm, and $\log^{(j)}$ is its j-fold iterate. For a graph G, an *even subgraph* is a set of edges that meets every vertex an even number of times. With symmetric difference as addition, the even subgraphs form a vector space $\mathcal{Z}(G)$ over $\mathbb{F}_2$. Its dimension is

$$\beta(G) = |E(G)| - |V(G)| + \mathrm{comp}(G),$$

where $\mathrm{comp}(G)$ is the number of components. We call $\beta(G)$ the *rank* of G; other names are cyclomatic number and cycle rank. A *cycle* is a connected 2-regular subgraph. We identify a subgraph without isolated vertices with its edge set. Then a cycle is a nonzero element of $\mathcal{Z}(G)$, so a graph of rank r has at most $2^r - 1$ cycles.

Let $\psi(r)$ be the largest number of cycles in a graph of rank r.

- Entringer and Slater [3] proved that $2^{r-1} < \psi(r) < 2^r$ for $r \geq 3$, and they conjectured that $\psi(r) \sim 2^{r-1}$.
- Aldred and Thomassen [2] proved that $\psi(r) \leq \frac{15}{16}2^r$.
- Alahmadi, Aldred, dela Cruz, Solé and Thomassen [1] proved that a connected graph with p vertices and more than $2p + 3\log(3p)$ edges has at most 2^{r-1} cycles.
- Horak and Kim [5] found cubic graphs of rank r with $H(r)$ cycles, where

$$H(2q+1) = 2^{2q} + \left(2q + \tfrac{1}{2}\right)2^q - 3q, \qquad H(2q+2) = 2^{2q+1} + \left(2q + \tfrac{9}{2}\right)2^q - 3q - 4. \tag{1.1}$$

For odd rank $r = 2q + 1$ the graph of Horak and Kim is a ring of q copies of $K_{2,2}$, each joined to the next by two edges. For even rank, one more edge is inserted across the ring (Figure 1). We call this graph R_r.

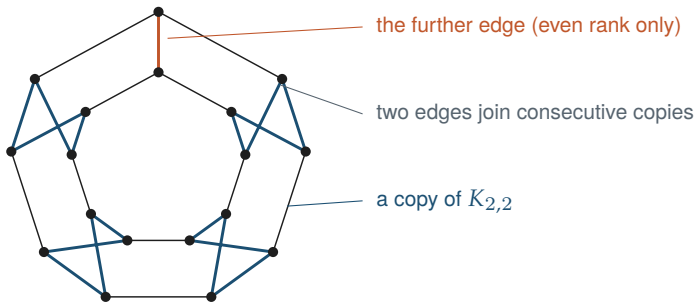


Figure 1. The graph R_{10}: a ring of four copies of $K_{2,2}$ and one further edge. It has rank 10 and $H(10) = 696$ cycles.

Theorem 1.1 (the maximum number of cycles)

There is an integer R such that $\psi(r) = H(r)$ for every $r \geq R$.

In particular $\psi(r) = 2^{r-1} + (1 + o(1))\, r\, 2^{\lfloor (r-1)/2 \rfloor}$, so $\psi(r) \sim 2^{r-1}$. The proof gives no value for R.

Fine print. The theorem is false for small r. For example, take the vertices u_i, v_i with $i \in \mathbb{Z}/26$ and the edges u_iu_{i+1}, u_iv_i and v_iv_{i+10}. This generalised Petersen graph has rank 27, and a direct count by computer gives 67 959 973 cycles, while $H(27) = 67\,325\,913$.

1.2 The three cases of the proof

The number of cycles does not change when we delete a vertex of degree at most one, or replace a vertex of degree two and its two edges by one edge. A vertex of degree at least four can be split into vertices of degree three, and no cycle is lost. So it suffices to count the cycles of a connected multigraph G of rank r in which all vertices have degree three (Lemma 3.1). The graphs R_r show that $\psi(r) \geq H(r)$.

For the upper bound there are three cases (Figure 2). A *cyclic cut* is a set of edges whose deletion leaves two parts that each contain a cycle. The number $H(r) - 2^{r-1}$ is of order $r\,2^{r/2}$, so the three bounds give the theorem (Section 8).

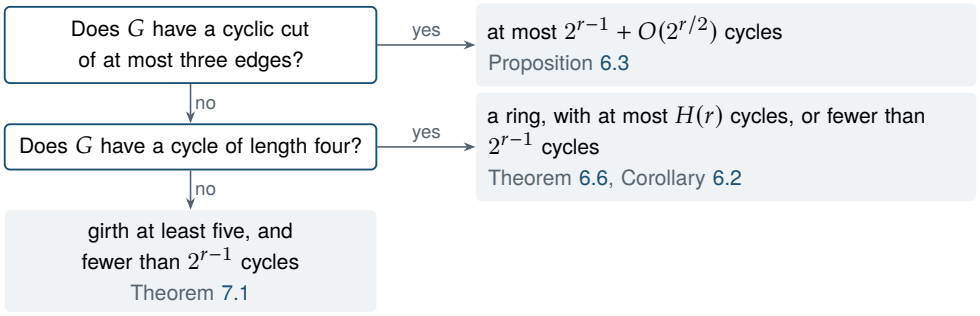


Figure 2. The three cases for a connected cubic multigraph G of large rank r.

1.3 Outside parts and the forest estimate

Fix a set P of vertices of G. For an even subgraph ξ we call the set of its edges without an end in P its *outside part* (Figure 3). If a cycle meets P, then its outside part is a proper subset of a cycle, so it is a forest.

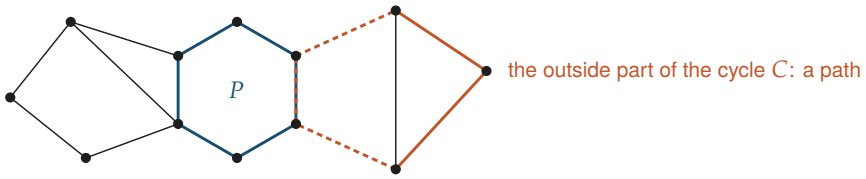


Figure 3. A set P of six vertices and a cycle C that meets P. The edges of C with an end in P are dashed. The other edges of C form its outside part, which is a path. An even subgraph whose outside part contains a cycle is not a cycle that meets P.

IDEA | cycles and outside parts

Let P be a small set of vertices such that $G[P]$ is connected. By the *forest estimate* of the companion paper [4] (Theorem 2.1), the outside part is a forest for at most $\frac{1}{2} + o(1)$ of all even subgraphs. So at most $(\frac{1}{2} + o(1))2^r$ cycles meet P. A cycle that avoids P is a cycle of $G - P$, and if G has no bridge, the rank of $G - P$ is at most $r - 1 - |P|/2$ (Lemma 3.2). This

gives $\psi(r) \le (\frac{1}{2} + o(1))2^r$ (Corollary 3.3).

In Cases 1 and 2 a cycle consists of paths in the parts of G. By the forest estimate, a part with more than half of the possible paths between two of its ends, by a fixed margin, is a ladder up to a bounded error, and it has few cycles (Section 5). In Case 2 we also put a small tree, a *cap*, in the place of the cycle of length four, and apply the bound $(\frac{1}{2} + o(1))2^r$ to the graph that arises (Lemma 6.4). In Case 3 the forest estimate is not sufficient, because we need a bound below $\frac{1}{2}$ and its error term is positive. In this case a second-moment argument gives two disjoint pieces of G with small boundaries, and each piece contributes a factor of at most $\frac{3}{5}$ to the proportion of forests (Section 7).

1.4 Authorship and verification

KNT is a person. Astra and Fable 5.2 are the language models GPT-6 Astra and Claude Fable 5.2. KNT directed the work at each step. KNT obtained the forest estimate in September 2026 with Astra, for a problem of Erdős on pancyclic graphs [6]; its proof is in [4]. For the number of cycles, Astra found the path structure of Section 5, the gap theorem, the count for rings, the cap, and the first proof of the last case. Fable 5.2 found the lemma on layers, the chain of separators and the pieces at girth five, and wrote this text. The present forms of the proofs contain steps of both models.

Theorem 1.1 has a formal proof in Lean 4. The statement `Erdos1016.ExactMaximum.exactMaximum_threeCases` says that there is a number R such that `maxCyclesr=horakKimr` for every $r \ge R$. Here `maxCyclesr` is the largest number of cycles of a simple graph with cyclomatic number r, and `horakKimr` is $H(r)$. The Lean kernel accepts the proof with the axioms `propext`, `Classical.choice` and `Quot.sound` only. The files are in [7], at commit `b7f9fc9`.

The formal proof has the three cases of Section 8, and it proves Theorem 7.1 at girth five. Some of its steps are not those of the text: a cut of size four has two arbitrary sides, Lemma 7.4 has a proof without ladders, and the end of the proof of Theorem 7.1 uses short cycles at many scales. So the kernel check covers the theorem, and it does not cover the text of this paper. No referee has read the text.

2 The forest estimate and its tools

This section states the five results of the companion paper [4] that we use. Their proofs are in [4].

NOTATION | multigraphs, cuts, regions and outside parts

- From here on a graph may have loops and parallel edges. We say *multigraph* to stress this, and *simple graph* to exclude it. A loop adds two to the degree of its vertex. A loop is a cycle of length one, and two parallel edges form a cycle of length two.
- A multigraph is *cubic* if every vertex has degree three, and *subcubic* if every vertex has degree at most three. The *girth* is the length of a shortest cycle.

- $c(G)$ is the number of cycles of G. If we replace an edge by a path, the rank and the number of cycles do not change. So $\psi(r)$ is also the largest number of cycles of a multigraph of rank r.
- For a set A of vertices, $G[A]$ is the induced subgraph, $G - A$ is $G[V(G) \setminus A]$, and δA is the set of edges with exactly one end in A.
- A *region* is a nonempty set A such that $G[A]$ is connected. A region is *cyclic* if $G[A]$ contains a cycle.
- A *cyclic cut* is a set δX such that $G[X]$ and $G - X$ each contain a cycle; its *size* is $|\delta X|$.
- Throughout, ξ is a uniformly random element of $\mathcal{Z}(G)$. For a region A, the event O_A is the event $\xi \cap \delta A = \emptyset$.
- For a set P of vertices and an even subgraph ξ, the *outside part* of ξ is $\xi \cap E(G - P)$, the set of its edges without an end in P. We write $q(G, P)$ for the probability that the outside part of ξ is a forest.

Theorem 2.1 (Forest estimate, [4, Theorem 1.1]). There is r_0 such that the following holds for all $r \geq r_0$. Let Γ be a connected subcubic multigraph of rank r. Let P be a set of at most $2r/\log^2 r$ vertices such that every vertex outside P has degree three and $\Gamma[P]$ has at most $(\log r)/100$ components. Then

$$q(\Gamma, P) \leq \frac{1}{2} + \frac{20}{\log^{(3)} r}.$$

This paper uses the theorem only when $\Gamma[P]$ is connected.

Let A and B be disjoint regions of a connected multigraph G. A *link* between them is an edge with one end in A and the other in B, or a component of $G - (A \cup B)$ that is joined by an edge to A and by an edge to B (Figure 4). Let $\ell(A, B)$ be the number of links.

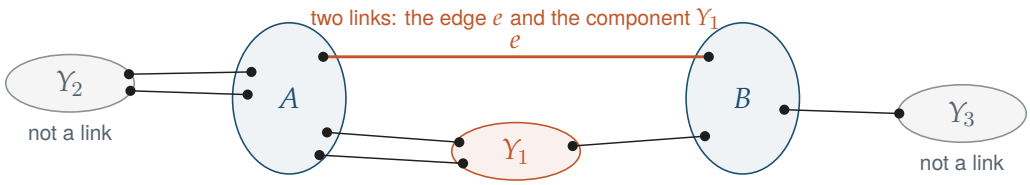


Figure 4. Links between two regions A and B. The edge e and the component Y_1 of $G - (A \cup B)$ are links. The components Y_2 and Y_3 are joined to one region only, so they are not links. Here $\ell(A, B) = 2$.

Lemma 2.2 (Link formula, [4, Lemma 2.2]). Let A and B be disjoint regions of a connected multigraph G. Then $\mathbb{P}(O_A) = 2^{\mathrm{comp}(G-A)-|\delta A|} \geq 2^{-|\delta A|}$ and

$$\mathbb{P}(O_A \cap O_B) = 2^{\ell(A,B)-1}\,\mathbb{P}(O_A)\,\mathbb{P}(O_B).$$

Conditionally on O_A, the set $\xi \cap E(G[A])$ is uniform on $\mathcal{Z}(G[A])$. Conditionally on $O_A \cap O_B$, the sets $\xi \cap E(G[A])$ and $\xi \cap E(G[B])$ are independent and uniform on $\mathcal{Z}(G[A])$ and $\mathcal{Z}(G[B])$.

So the events O_A and O_B are independent when the regions have one link, and their joint probability is twice the product when the regions have two links.

Lemma 2.3 (Exceptional partners, [4, Lemma 2.3]). Let A be a region of a connected multigraph G, and let $B_1, \dots, B_k$ be pairwise disjoint regions, disjoint from A, with $\ell(A, B_i) \geq 3$ for all i. Then $k \leq |\delta A|$.

Lemma 2.4 (Many small cuts, [4, Lemma 2.4]). Let $A_1, \dots, A_k$ be pairwise disjoint cyclic regions of a connected multigraph G with $|\delta A_i| \leq d$ for all i. Then

$$\mathbb{P}\big(\xi \cap E(G[A_i]) \text{ is a forest for every } i\big) \;\leq\; \frac{1}{2} + \frac{(d+1)\,2^{d-1}}{k}.$$

The last result is on the cycles of moderate length that remain when a maximal family of short cycles is deleted. Section 7.5 uses it.

Proposition 2.5 (Cycles of moderate length, [4, Proposition 3.4]). There is r_2 such that the following holds for all $r \geq r_2$, with $x = \log r$, $y = \log x$ and $\eta = r^{-1/60}$. Let Γ and P be as in Theorem 2.1, with $q(\Gamma, P) \geq \frac{2}{5}$. Let D and L be integers with $x/16 \leq D \leq x/8$ and $7x \leq L \leq x\sqrt{y}$. Let S be the union of the vertex sets of a maximal family of pairwise disjoint cycles of length at most D in $\Gamma - P$, and suppose that $|S| \leq r/x^2$. Put $J = \Gamma - (P \cup S)$. Let $\mathcal{C}$ be the set of the cycles of J of length at most L, with the weights $a_C = 2^{-|C|}$ of total μ. Let W_2 be the sum of $a_C a_{C'}$ over the ordered pairs of disjoint cycles $C, C' \in \mathcal{C}$ such that $V(C)$ and $V(C')$ have exactly two links in Γ. Then

(i) J is a simple graph of girth greater than D;

(ii) for every vertex w of J, the cycles of $\mathcal{C}$ through w have total weight at most η/L;

(iii) $\mu \geq \frac{1}{4} \ln \dfrac{L}{6x} - \dfrac{1}{\sqrt{x}}$;

(iv) some random variable $Z \geq 0$ has $\mathbb{E}Z = \mu$ and $\mathbb{E}Z^2 \leq \mu^2 + W_2 + (1+\eta)\,\mu$, and $Z = 0$ whenever ξ contains no cycle of J. Here ξ is uniform on $\mathcal{Z}(\Gamma)$.

In [4] the variable Z is a weighted number of the cycles of C that are components of ξ.

3 The leading term

This section proves that $\psi(r) \le (\frac{1}{2} + o(1))\, 2^r$. The later sections use this bound and the two lemmas.

Lemma 3.1 (Cubic multigraphs). For every $r \ge 2$ there is a connected cubic multigraph of rank r with exactly $\psi(r)$ cycles.

Proof. The numbers of cycles of the graphs of rank r are integers below 2^r, so some graph G_0 attains $\psi(r)$. If G_0 is not connected, identify a vertex of one component with a vertex of another. This changes neither the rank nor the set of cycles. Now delete a vertex of degree one, or replace a vertex of degree two and its two edges by one edge, as long as this is possible. These steps keep the rank and the number of cycles. The result is a connected multigraph with all degrees at least three, because $r \ge 2$.

Let v be a vertex of degree $d \ge 4$. Replace v by a path $v^1 v^2 \cdots v^{d-2}$, and distribute the d edge ends at v along this path: two at v^1, one at each of $v^2, \dots, v^{d-3}$, and two at v^{d-2}. The new multigraph has $d-3$ more vertices and $d-3$ more edges, so its rank is r. A cycle through v uses two edge ends at v. Together with the part of the path between these two ends it is a cycle of the new multigraph. Different cycles give different cycles, and the cycles that avoid v remain. So the number of cycles does not decrease, and it cannot exceed $\psi(r)$. Repeat this for every vertex of degree at least four. ∎

Lemma 3.2 (Cycles and a marked set). Let G be a connected cubic multigraph of rank r without a bridge. Let P be a set of p vertices, not all vertices of G, such that $G[P]$ is connected. Then

$$c(G) \le q(G,P)\, 2^r + 2^{r-1-p/2}.$$

Proof. Let Z be a cycle of G that meets P. Its edges in $G - P$ form a proper subset of Z, hence a forest. So at most $q(G,P)\, 2^r$ cycles meet P. The other cycles are cycles of $G - P$, and there are at most $2^{\beta(G-P)}$ of them. Let a be the number of edges of $G[P]$. Then $3p - 2a$ edges leave P, and $G - P$ arises from G by the deletion of p vertices and $3p - a$ edges. So $\beta(G-P) = r-1-2p+a+\mathrm{comp}(G-P)$. The graph G has no bridge, so every component of $G-P$ contains the ends of at least two of the edges that leave P. Hence $\mathrm{comp}(G-P) \le (3p-2a)/2$ and $\beta(G-P) \le r - 1 - p/2$. ∎

Corollary 3.3 (Leading term). There is r_1 such that every graph of rank $r \ge r_1$ has at most $\big(\frac{1}{2} + 21/\log^{(3)} r\big)2^r$ cycles. Consequently $\psi(r) \sim 2^{r-1}$.

Proof. Let r be large, and let G be a connected cubic multigraph of rank r with $\psi(r)$ cycles (Lemma 3.1). Suppose first that G has a bridge e, and let b_1 and b_2 be the ranks of the two components of $G - e$. A component with n vertices has $(3n-1)/2 \geq n$ edges, so $b_1, b_2 \geq 1$, and $b_1 + b_2 = r$. Every cycle lies in one component. So $c(G) \leq (2^{b_1} - 1) + (2^{b_2} - 1) \leq 2^{r-1}$.

In the other case choose a set P of $\lceil 4 \log r \rceil$ vertices such that $G[P]$ is connected. Then $|P| \leq 2r/\log^2 r$. By Theorem 2.1 and Lemma 3.2,

$$c(G) \;\leq\; \Big(\frac{1}{2} + \frac{20}{\log^{(3)} r}\Big)2^r + 2^{\,r-1-2\log r} \;\leq\; \Big(\frac{1}{2} + \frac{21}{\log^{(3)} r}\Big)2^r.$$

Finally $\psi(r) > 2^{r-1}$ for $r \geq 3$ [3]; the rings of Section 6.1 also show this. ∎

4 Layers

The later sections use Theorem 2.1 in two ways: by Corollary 3.3, and by the lemma of this section.

IDEA | the content of the lemma

If the outside part is a forest for more than half of the even subgraphs, by a fixed margin, then every distance layer around the marked set is small. In the other case a window of consecutive layers is a large set with a small boundary, and the forest estimate applies inside this set. We use the lemma for paths in Section 5 and for pieces in Section 7.

Let G be a multigraph and P a set of vertices. A *level function* is a map $\lambda : V(G) \to \mathbb{Z}$ such that $|\lambda(u) - \lambda(w)| \leq 1$ for every edge uw that has an end outside P. The distance from a fixed vertex is a level function. For $\delta \in (0, \frac{1}{2})$ let $R_\delta \geq 2^{100}$ be an integer such that every $r \geq R_\delta$ is in the range of Theorem 2.1 and satisfies $20/\log^{(3)} r \leq \delta$. For an integer $k \geq 1$ put

$$\Theta_\delta(k) = \max\big\{2R_\delta,\; 2^{100}(k+1)^2\big\}.$$

Lemma 4.1 (Layers). Let G be a connected subcubic multigraph, and let P be a nonempty set of vertices such that $G[P]$ is connected and every vertex outside P has degree three. Let λ be a level function such that at most k vertices of P have any given level. Let $\delta \in (0, \frac{1}{2})$, and suppose that $q(G, P) > \frac{1}{2} + \delta$. Then, for every j, fewer than $\Theta_\delta(k)$ vertices outside P have level j.

Proof. Put $\Phi(m) = m/\log^2 m$ and $\Theta = \Theta_\delta(k)$.

Step 1: no large set outside P has a small boundary. We claim that no set $X \subseteq V(G) \setminus P$ has $|X| = m \geq \Theta$ and $|\delta X| \leq \Phi(m)/4$. Suppose that X is such a set. Let C_0 be the component of $G - X$ that contains P, and put $X^+ = V(G) \setminus V(C_0)$. Then $X \subseteq X^+$, the set X^+ avoids P, and $G - X^+ = C_0$ is connected. Different components of $G - X$ have no edge between them, so $\delta X^+ \subseteq \delta X$. Every vertex of X^+ has degree three. Put $d^+ = |\delta X^+| \geq 1$.

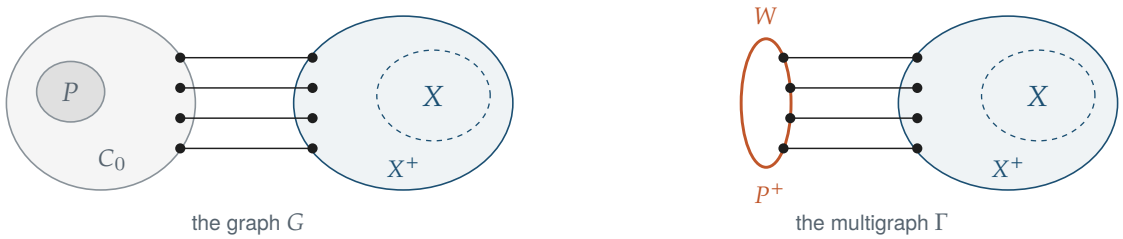


Figure 5. Step 1 in the proof of Lemma 4.1. In G the set X^+ is joined by d^+ edges to the connected rest C_0, which contains P. In Γ a cycle W through d^+ new vertices takes the place of C_0. The even subgraphs of G and of Γ give the same distribution of the edges inside X^+.

We form a multigraph Γ from $G[X^+]$ (Figure 5). For each edge of δX^+ add a new vertex and join it to the end of that edge in X^+. Join the d^+ new vertices by a cycle W through all of them, and let P^+ be the set of the new vertices. Every component of $G[X^+]$ has an edge to C_0, so Γ is connected. It is subcubic, $\Gamma[P^+]$ is connected, the vertices outside P^+ have degree three, and $\Gamma - P^+ = G[X^+]$. The multigraph Γ has d^+ more vertices and $2d^+$ more edges than $G[X^+]$, and $3|X^+| = 2|E(G[X^+])| + d^+$. So its rank is

$$\rho = \frac{|X^+| + d^+}{2} + 1 \;\geq\; \frac{m}{2} \;\geq\; R_\delta.$$

The function Φ is increasing on $[8, \infty)$, and $\Phi(m)/4 \leq \Phi(m/2)$. So $d^+ \leq \Phi(m)/4 \leq \Phi(m/2) \leq \Phi(\rho)$, and Theorem 2.1 with the choice of R_δ gives $q(\Gamma, P^+) \leq \frac{1}{2} + \delta$.

The set $\xi \cap E(G[X^+])$ has the same distribution for ξ uniform on $\mathcal{Z}(G)$ and for ξ uniform on $\mathcal{Z}(\Gamma)$. Indeed, contract the connected set $V(C_0)$ of G to one vertex, and delete the loops that arise. Call the result G'. An even subgraph of G restricts to an even subgraph of G'. Every even subgraph of G' arises in this way: the missing edges inside the contracted set can be chosen in a spanning tree of that set. So the restriction is a surjective linear map $\mathcal{Z}(G) \to \mathcal{Z}(G')$, and it sends the uniform distribution to the uniform distribution. The contraction of W in Γ gives the same multigraph G', with the new edges in the place of the edges of δX^+. Hence

$$\mathbb{P}\big(\xi \cap E(G[X^+]) \text{ is a forest}\big) \;=\; q(\Gamma, P^+) \;\leq\; \frac{1}{2} + \delta.$$

But $G[X^+]$ is a subgraph of $G - P$, so this probability is at least $q(G, P) > \frac{1}{2} + \delta$. This contradiction proves the claim.

Step 2: a window of levels. Let m_j be the number of vertices outside P of level j. For $i \leq i'$ let $X(i, i')$ be the set of vertices outside P with a level in $[i, i']$. An edge that leaves $X(i, i')$ ends outside P at level $i - 1$ or $i' + 1$, or it ends in P at a level in $[i - 1, i' + 1]$. So

$$|\delta X(i, i')| \;\leq\; 3m_{i-1} + 3m_{i'+1} + 3k(i' - i + 3). \tag{4.1}$$

Suppose that $m_j \geq \Theta$ for some j. Start with the window $i = i' = j$. As long as a level next to the window has more than $m^{3/4}$ vertices outside P, where $m = |X(i, i')|$ is the present size, add that level to the window. The graph is finite, so this stops. A step increases $m^{1/4}$

by more than $\frac{1}{6}$, because $(t+\frac{1}{6})^4 \le t^4 + t^3$ for $t \ge 1$. Let μ be the final size. Then there were fewer than $6\mu^{1/4}$ steps, and both levels next to the final window have at most $\mu^{3/4}$ vertices outside P. We have $k \le \sqrt{\mu}$, because $\mu \ge \Theta \ge 2^{100}(k+1)^2$. By (4.1),

$$|\delta X(i,i')| \;\le\; 6\mu^{3/4} + 3k\bigl(6\mu^{1/4}+3\bigr) \;\le\; 33\,\mu^{3/4} \;\le\; \frac{\mu}{4\log^2\mu} = \frac{\Phi(\mu)}{4}.$$

The last inequality says that $132\log^2\mu \le \mu^{1/4}$, and this holds for $\mu \ge 2^{100}$. So the final window contradicts Step 1. ∎

5 Poles with many paths

A cut of a cubic graph divides it into two parts, and a cycle that crosses the cut is made of paths in the two parts. So we need to count paths between prescribed ends.

IDEA | the results of this section

First, if more than half of the possible paths between two ends are present, by a fixed margin, then the part looks like a ladder, and it has few cycles (Theorem 5.5). Second, the number of paths between two ends is the largest possible, or it is smaller by a fixed factor (Theorem 5.7): a part that is not a ladder has a short cycle away from a shortest path between the two ends.

5.1 Poles

A *pole* is a connected multigraph without loops in which every vertex has degree two or three. Its vertices of degree two are its *ports*. A *k-pole* is a pole with exactly k ports. By a count of edge ends, a k-pole of rank h has $2h+k-2$ vertices.

For two vertices $s \ne t$ of a multigraph N let $P_{st}(N)$ be the number of paths from s to t. An *$\{s,t\}$-join* is a set of edges in which s and t have odd degree and all other vertices have even degree. If N is connected of rank b, there are exactly 2^b of them: they exist, and the sum of two of them is even. Every path from s to t is an $\{s,t\}$-join, so

$$P_{st}(N) \;\le\; 2^b. \tag{5.1}$$

Call the pair s,t *full* in N if $P_{st}(N) = 2^{\beta(N)}$.

Lemma 5.1 (Joins). Let N be a pole of rank b, and let s,t be two of its ports. Every $\{s,t\}$-join is the union of a path from s to t and of cycles, and these are pairwise vertex-disjoint. So the pair is full if and only if every cycle of N meets every path from s to t. If some cycle of N meets no path from s to t, then $P_{st}(N) \le 2^{b-1}$. If a cycle Z of N is disjoint from some path from s to t, then $P_{st}(N) \le 2^b - 2^{b-|Z|}$.

Proof. In a join, s and t have odd degree at most two, and every other vertex has even degree at most three. So the degrees are one at s and t and zero or two elsewhere. If a cycle Z meets no path from s to t, then $J \mapsto J + Z$ maps the paths injectively to joins that are not paths. Let a cycle Z be disjoint from a path Q from s to t. Then Q is an $\{s,t\}$-join of $N - E(Z)$. So $N - E(Z)$ has $2^{\beta(N-E(Z))}$ such joins, and $\beta(N - E(Z)) \geq b - |Z|$. The union of each of them with $E(Z)$ is an $\{s,t\}$-join of N that is not a path. ■

Lemma 5.2 (Reduction). Let N be a pole of rank b, and let s, t be two of its ports with $P_{st}(N) > 2^{b-1}$.

(a) Replace each port other than s, t, with its two edges, by one edge. No loop arises. The result N° is a 2-pole with ports s, t. It has the same rank, the same number of paths from s to t, and the same number of cycles as N.

(b) Every bridge of N° separates s from t and is incident with neither. Let $N_1, \dots, N_q$ be the components of N° without its bridges, in the order from s to t. Each N_i is a bridgeless 2-pole of rank $b_i \geq 1$, and

$$b = \sum_i b_i, \qquad P_{st}(N) = \prod_i P(N_i), \qquad c(N) = \sum_i c(N_i),$$

where $P(N_i)$ is the number of paths between the two ports of N_i. In particular $P(N_i)\, 2^{-b_i} \geq P_{st}(N)\, 2^{-b}$ for every i.

Proof. (a) Let v be a port other than s, t. Suppose that both edges at v end in the same vertex u. Then u has degree three, its third edge is a bridge, and no path from s to t meets the cycle through u and v. This contradicts Lemma 5.1. So the two neighbours of v are distinct, and we may replace v by an edge. Paths and cycles correspond. Repeat.

(b) Let e be a bridge of N°, and let Y be a component of $N^\circ - e$, with n vertices, j of them ports. Counting edge ends gives $\beta(Y) = (n - j + 1)/2$. This is a positive integer, so Y contains a cycle. If Y contains neither port or both ports, a cycle on one side of e meets no path from s to t. If e is incident with s, every path from s to t starts with e and avoids the cycle on the side of s. Both cases contradict Lemma 5.1. So every path from s to t passes all bridges, in the same order.

The multigraph N_i is connected and bridgeless. Its vertices have degree three, except the ends of the two bridges at N_i, or s, or t. These two vertices are distinct: a common end of two bridges would have a third edge that is a bridge and does not separate s from t. So N_i is a 2-pole, and it contains a cycle. Ranks add over the N_i, every cycle lies in one N_i, and a path from s to t is a concatenation of paths between the ports of the N_i and of the bridges. The last statement holds because $P(N_j) \leq 2^{b_j}$ for all j. ■

5.2 A shortest path between the ports

Let N be a bridgeless 2-pole with ports s, t and rank $b \geq 2$. It has $2b$ vertices. It has no cut vertex: a cut vertex of a subcubic multigraph is joined to some component by a single edge. We use the following notation. The *layer* L_j is the set of vertices at distance j from s. Every edge joins vertices of the same layer or of consecutive layers. Let ℓ be the distance from s to t, let $A = w_0 w_1 \cdots w_\ell$ be a shortest path from s to t, and put

$$J = N - V(A), \qquad \Delta = b - \ell.$$

Then $w_j \in L_j$. Write e_i for the edge $w_{i-1}w_i$ of A. A vertex of A lies on two edges of A, or on one if it is a port. So it has at most one *further edge*. This edge leads to J, or it is parallel to an edge of A, because A is a shortest path. For $0 < j < \ell$ the layer L_j separates s from t, so $|L_j| \geq 2$. Call L_j *thin* if $0 < j < \ell$ and $|L_j| = 2$. A thin layer consists of w_j and one vertex of J (Figure 6).

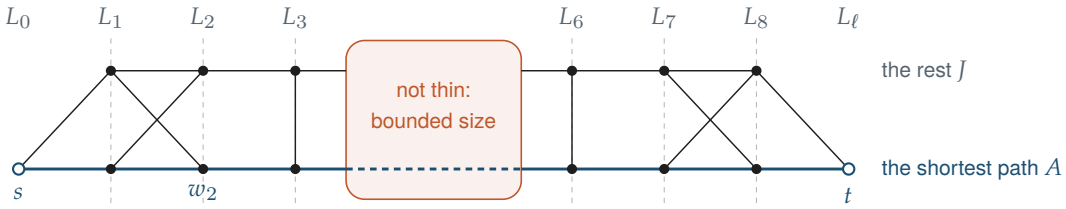


Figure 6. A bridgeless 2-pole with many paths from s to t (schematic). Almost every layer is thin: it has one vertex of the shortest path A and one vertex of J (Lemma 5.3 and Proposition 5.4).

IDEA | the content of Lemma 5.3

The lemma has no hypothesis on the number of paths. It compares N with a pole that has exactly two vertices in each layer between the ports; N has 2Δ vertices more. So at most $6\Delta + 2$ vertices of N lie outside the thin layers, and Δ is at most the rank of J. Between two thin layers a cycle has at most two ways to continue (Figure 7). Statement (d) counts the cycles by their edges on A. It follows that a pole in which J has a small rank has few cycles.

Lemma 5.3 (A shortest path). Let N be a bridgeless 2-pole of rank $b \geq 2$. In the notation above,

(a) $0 \leq \Delta \leq \beta(J)$;

(b) at most 2Δ of the layers L_j with $0 < j < \ell$ are not thin, and at most 4Δ vertices of J lie outside the thin layers;

(c) every cycle of J has at most $4\Delta + 2$ vertices;

(d) $c(N) \leq 10 \cdot 2^{5\beta(J)+b/2}$.

Proof. (a) The layers $L_0, \ldots, L_\ell$ contain at least 2ℓ vertices, so $\ell \leq b$. The graph J has $2b - \ell - 1 \geq 1$ vertices. The vertices of A meet the ℓ edges of A and at most $\ell + 1$ other edges, and N has $3b - 1$ edges. So $|E(J)| \geq (3b - 1) - (2\ell + 1)$, and $\beta(J) \geq |E(J)| - |V(J)| + 1 \geq b - \ell$.

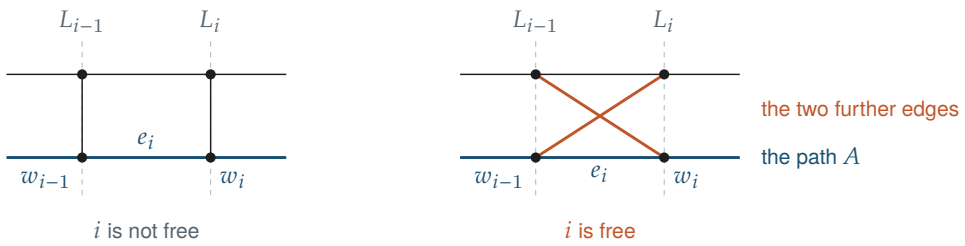


Figure 7. Two thin layers L_{i-1} and L_i. Left: the index i is not free, and a cycle that passes both layers contains the edge e_i of A. Right: the index i is free. A cycle can contain e_i, or it can contain the further edges of w_{i-1} and of w_i.

(b) Compare N with a multigraph that has one vertex in L_0 and in L_ℓ, two in each layer between, and none beyond. Then N has exactly 2Δ vertices more. A layer between L_0 and L_ℓ with $n \geq 3$ vertices has $n-2$ vertices too many, and $n-1 \leq 2(n-2)$ of its vertices are in J. Each vertex of L_ℓ other than t, and each vertex beyond L_ℓ, is one vertex too many, and it is in J. This gives (b).

(c) Let Z be a cycle of J, and let L_i and L_j be the first and the last layer that it meets. A thin layer has one vertex of J. If $i < l < j$, the two paths of Z between a vertex in L_i and a vertex in L_j both have a vertex in L_l, so L_l is not thin. Hence at most two vertices of Z are in thin layers, and (b) gives (c).

(d) An even subgraph of N without an edge of A has degree at most one at each vertex of A, hence degree zero. So it is an even subgraph of J, and there are $2^{\beta(J)}$ of them. The sum of two even subgraphs with the same edges in A has no edge of A. So at most $2^{\beta(J)}$ cycles of N have the same edges in A.

Let Z be a cycle that contains an edge of A, and let e_a and e_z be the first and the last edge of A in Z.

Two strands. Let $L_l = \{w_l, y_l\}$ be a thin layer with $a \leq l < z$. The cycle Z contains w_{a-1} and w_z, so it has vertices before and after this layer. So it has at least two edges from L_{l-1} to L_l and at least two edges from L_l to L_{l+1}. The two vertices of L_l have degree at most two in Z. So Z has exactly two edges on each side of L_l. Suppose that both edges from L_{l-1} end in the same vertex of L_l. Then both edges to L_{l+1} start in the other vertex, and no edge of Z joins the part of Z before L_l to the part after it. This is impossible, because Z is connected. So each vertex of L_l has one edge of Z to L_{l-1} and one to L_{l+1}.

Free indices. Call an index i *free* if the further edge of w_{i-1} leads to L_i and the further edge of w_i leads to L_{i-1} (Figure 7). Two consecutive indices are not both free, because w_i has one further edge at most. Let $a < i < z$, and suppose that the layers L_{i-1} and L_i are thin and that e_i is not in Z. By the last paragraph, Z has an edge from w_{i-1} to L_i and an edge from w_i to L_{i-1}. These are further edges. So i is free.

The count. So the edges of Z in A are determined by a and z and by a choice for each index i with $a < i < z$ that is free, or for which L_{i-1} or L_i is not thin. At most $(z-a)/2$ of the indices between a and z are free, and by (b) at most 4Δ indices are of the second kind. So for given

$a \le z$ there are at most $2^{4\Delta+(z-a)/2}$ possibilities. With the first paragraph,

$$c(N) \;\le\; 2^{\beta(J)}\Big(1 + 2^{4\Delta} \sum_{1\le a\le z\le \ell} 2^{(z-a)/2}\Big) \;\le\; 2^{\beta(J)}\big(1 + 9\cdot 2^{4\Delta+\ell/2}\big) \;\le\; 10\cdot 2^{5\beta(J)+b/2}.$$

Here the sum is at most $2^{\ell/2}\sum_{i\ge 1} i\, 2^{-i/2} = (4+3\sqrt{2})\, 2^{\ell/2}$, and we use $\Delta \le \beta(J)$ and $\ell \le b$. ■

IDEA | outline of the proof of Proposition 5.4

Add the shortest path A to a path from s to t. The result is an even subgraph whose edges away from A form a forest. So the outside part is a forest for more than half of the even subgraphs, and every layer is small by Lemma 4.1. By Lemma 2.4 there are few short cycles away from A. So the rank of J is bounded. By Lemma 5.3, almost every layer then has exactly two vertices, and N has few cycles.

Proposition 5.4 (Away from the shortest path). For every $\varepsilon \in (0, \frac{1}{2}]$ there is a number $C = C_\varepsilon \ge 1$ with the following property. Let N be a bridgeless 2-pole of rank $b \ge 2$ with ports s, t and $P_{st}(N) \ge (\frac{1}{2} + \varepsilon)2^b$. Then $\beta(J) \le C$, in the notation above.

Proof. Step 1: the layers are small. Put $P = V(A)$. Then $N[P]$ is connected, and every vertex outside P has degree three. For a path Q from s to t, the set $Q + A$ is an even subgraph of N. Its edges in $N - P$ are edges of the path Q, so they form a forest. Different paths give different even subgraphs, and N has 2^b even subgraphs. Hence

$$q(N,P) \;\ge\; \frac{P_{st}(N)}{2^b} \;\ge\; \frac{1}{2} + \varepsilon. \tag{5.2}$$

Let $\lambda(u)$ be the distance from s to u in N. This is a level function, and P has at most one vertex of each level. By Lemma 4.1 with $\delta = \varepsilon/2$, every layer L_j has fewer than $\Theta := \Theta_{\varepsilon/2}(1)$ vertices of J.

Step 2: a short cycle. Put $U_j = L_j \setminus V(A)$, and for an interval I of indices let J_I be the subgraph of J induced by the sets U_j with $j \in I$. Suppose that J_I contains a cycle. Choose a cycle Z of J_I such that the interval $[a, a']$ of the layers that Z meets is as short as possible, and put $\sigma = a' - a$ and $R = U_a \cup \cdots \cup U_{a'}$. We claim that $|R| \le 20\Theta$.

By the choice of Z, every cycle of $J[R]$ meets U_a. So $J[R] - U_a$ is a forest. Deleting a vertex of degree at most three lowers the rank by at most two, so $\beta(J[R]) \le 2|U_a| < 2\Theta$. The vertices of R have degree three in N. Counting edge ends gives $|R| = 2\beta(J[R]) + |\delta R| - 2\,\mathrm{comp}(J[R])$. An edge that leaves R ends in U_{a-1}, in $U_{a'+1}$, or in one of the vertices $w_{a-1}, \dots, w_{a'+1}$, each of which has at most one further edge. So $|\delta R| < 6\Theta + \sigma + 3$, and $|R| < 10\Theta + \sigma + 1$. The cycle Z has a vertex in layer a and a vertex in layer a', so $|Z| \ge 2\sigma$. Also $|Z| \le |R|$. Hence $\sigma \le 10\Theta$ and $|R| \le 20\Theta$.

Step 3: the rank of J. Start with the interval of all layers. As long as J_I contains a cycle for

some interval I of the present family, choose Z and R as in Step 2, and replace I by the at most two intervals that remain when $[a, a']$ is removed. Let $Z_1, \dots, Z_k$ be the chosen cycles and $R_1, \dots, R_k$ the corresponding sets. The sets R_i are pairwise disjoint, $|R_i| \le 20\Theta$, and $J - \bigcup_i R_i$ is a forest, because an edge of J joins equal or consecutive layers. So $\beta(J) \le 2\sum_i |R_i| \le 40\Theta k$.

The sets $V(Z_i)$ are pairwise disjoint cyclic regions of N that avoid P. Each has at most $|Z_i| \le 20\Theta$ edges that leave it. If $\xi \cap E(N - P)$ is a forest, then $\xi \cap E(N[V(Z_i)])$ is a forest for every i. So Lemma 2.4 and (5.2) give $k \le (20\Theta + 1)2^{20\Theta - 1}/\varepsilon$. Hence $\beta(J) \le C$ with

$$C = C_\varepsilon := \frac{40\Theta\,(20\Theta + 1)\,2^{20\Theta - 1}}{\varepsilon}.$$ ■

Theorem 5.5 (Few cycles). For every $\varepsilon \in (0, \frac{1}{2}]$ there is a constant K_ε with the following property. Let N be a pole of rank b, and let s, t be two of its ports with $P_{st}(N) \ge (\frac{1}{2} + \varepsilon)2^b$. Then

$$c(N) \;\le\; K_\varepsilon\, 2^{b/2}.$$

Proof. In the notation of Lemma 5.2, $c(N) = \sum_i c(N_i)$, and each N_i is a bridgeless 2-pole with at least $(\frac{1}{2} + \varepsilon)2^{b_i}$ paths between its ports. If $b_i = 1$, then $c(N_i) = 1$. If $b_i \ge 2$, then $c(N_i) \le 10 \cdot 2^{5C_\varepsilon}\, 2^{b_i/2}$ by Proposition 5.4 and Lemma 5.3(d). Also $\sum_i 2^{b_i/2} \le 2^{b/2} + q - 1 \le 2 \cdot 2^{b/2}$, because $x + y \le xy + 1$ for $x, y \ge 1$. So the theorem holds with $K_\varepsilon = 20 \cdot 2^{5C_\varepsilon}$. ■

5.3 Full pairs

A *square* is a copy of the complete bipartite graph $K_{2,2}$; its two classes are its *left pair* and its *right pair*. A *single rung* is one edge; its two ends form both its left pair and its right pair. These are the two kinds of *tiles*. The *weight* of a square is 2, and the weight of a single rung is 1. A *chain* of tiles $\tau_1, \dots, \tau_m$ consists of these tiles, pairwise disjoint, and of two disjoint edges from the right pair of τ_i to the left pair of τ_{i+1} for $1 \le i < m$. These two edges are an *interface*. A *capped ladder* is a chain with two more vertices s and t, the *caps*: s is joined to the two vertices of the left pair of τ_1, and t to the two vertices of the right pair of τ_m (Figure 8). A capped ladder is a 2-pole. If its tiles have total weight W, it has $2W + 2$ vertices and rank $W + 1$.

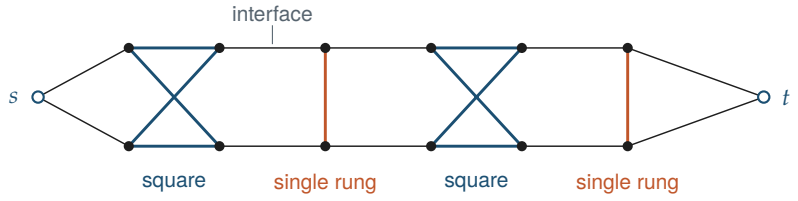


Figure 8. A capped ladder with four tiles, of weight 6 and rank 7. Without s and t it is a chain. If the right pair of the last tile is joined to the left pair of the first tile instead, the result is a ring (Section 6.1).

Lemma 5.6 (Ladders). Let N be a bridgeless 2-pole with ports s, t and rank b. If $b = 1$, then N consists of two vertices and two parallel edges, and the pair s, t is full. If $b \geq 2$, the following statements are equivalent, in the notation of Section 5.2:

(i) the pair s, t is full;

(ii) the graph J is a forest;

(iii) N is a capped ladder whose caps are s and t.

Proof. If $b = 1$, then N has two vertices, and it is a cycle. Let $b \geq 2$.

From (i) to (ii). By Lemma 5.1, every cycle meets the path A.

From (ii) to (iii). By Lemma 5.3(a), $b - \ell \leq \beta(J) = 0$. So $\ell = b$, and equality holds in every estimate of the proof of Lemma 5.3(a). This means the following. Every layer L_j with $0 < j < b$ consists of w_j and of one vertex y_j of J, and there are no other vertices outside A. Every vertex of A has a further edge, and this edge leads to J. And J is a tree.

An edge joins vertices of equal or consecutive layers. So J is the path $y_1 y_2 \cdots y_{b-1}$, the further edge of s leads to y_1, that of t leads to y_{b-1}, and the further edge of w_j leads to a vertex $y_{\pi(j)}$ with $|\pi(j) - j| \leq 1$, for $0 < j < b$. Every y_i has degree three, so it is the end of exactly one of the edges $w_j y_{\pi(j)}$. Hence π is a permutation of $\{1, \dots, b-1\}$ that moves every number by at most one. So π is composed of fixed points and of exchanges of two consecutive numbers. A fixed point j is a single rung with the ends w_j, y_j. An exchange of j and $j+1$ is a square with the left pair w_j, y_j and the right pair w_{j+1}, y_{j+1}. The remaining edges of A and J are the interfaces and the edges at the caps.

From (iii) to (i). Let the tiles have total weight W, so $b = W + 1$. A path from s to t passes the tiles in order. It enters a tile at a vertex of the left pair and leaves it at a vertex of the right pair. In a square there are two paths to each vertex of the right pair. In a single rung the path stays or crosses. So there are $2 \cdot 2^W = 2^b$ paths. ∎

Theorem 5.7 (Gap). There is a constant $\gamma_0 \in (0, \frac{1}{4}]$ with the following property. Let N be a pole of rank b, and let s, t be two of its ports. Then

$$P_{st}(N) = 2^b \qquad \text{or} \qquad P_{st}(N) \leq (1 - \gamma_0)\, 2^b.$$

Proof. Put $C = C_{1/4}$ and $\gamma_0 = 2^{-4C-2}$. Let $P_{st}(N) > (1 - \gamma_0)2^b$. In the notation of Lemma 5.2, the path counts of the bridgeless 2-poles $N_1, \dots, N_q$ multiply, and $P(N_i) > (1 - \gamma_0)2^{b_i}$ for every i. So it suffices to show that the pair of ports of each N_i is full.

Let N be one of them, with rank b. If $b = 1$, the pair is full. Let $b \geq 2$, and suppose that J contains a cycle Z. We have $P_{st}(N) \geq \frac{3}{4}\, 2^b$. So $\beta(J) \leq C$ by Proposition 5.4, and Z has at most $4C + 2$ vertices by Lemma 5.3(a) and (c). The cycle Z is disjoint from the path A. By Lemma 5.1, $P_{st}(N) \leq 2^b - 2^{b-|Z|} \leq (1 - \gamma_0)2^b$. This is a contradiction. So J is a forest, and

the pair is full by Lemma 5.6. ∎

Corollary 5.8 (Four ports). Let N be a 4-pole without bridge, with ports a, b, c, d such that the four pairs ac, ad, bc, bd are full. Then N is a chain of tiles; a, b is the left pair of its first tile and c, d is the right pair of its last tile.

Proof. Add a vertex z joined to a, b and a vertex w joined to c, d. The result N' is a 2-pole of rank $\beta(N) + 2$ with ports z, w, and $P_{zw}(N') = 4 \cdot 2^{\beta(N)}$. So the pair z, w is full. No edge of N is a bridge of N', because N has no bridge. No edge at z is a bridge, because the second edge at z ends in another vertex of N; the same holds for w. So N' has no bridge, and it has more than two vertices. By Lemma 5.6, N' is a capped ladder with the caps z and w. The neighbours a, b of z form the left pair of its first tile, and the neighbours c, d of w form the right pair of its last tile. Without the caps, this is a chain. ∎

5.4 Three gadgets

For four distinct ports a, b, c, d of a pole N let $f_{ab|cd}(N)$ be the number of pairs of vertex-disjoint paths of N, one from a to b and one from c to d.

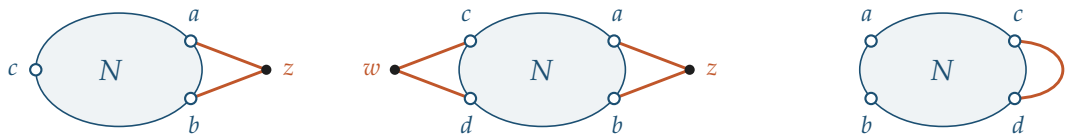


Figure 9. The three gadgets in the proof of Lemma 5.9: a new vertex z at the ports a and b; new vertices z and w; a new edge between c and d.

Lemma 5.9 (Three gadgets). Let $\varepsilon \in (0, \frac{1}{2}]$, and let N be a pole of rank h. In each statement below, the letters denote distinct ports of N. Write P_{xy} for $P_{xy}(N)$ and f for $f(N)$.

(a) If $P_{ab} \geq (\frac{1}{2} + \varepsilon)2^h$, then $c(N) \leq K_\varepsilon\, 2^{h/2}$.

(b) If $P_{ac} + P_{bc} \geq (1 + 2\varepsilon)2^h$, then $P_{ab} \leq 2K_\varepsilon\, 2^{h/2}$.

(c) If $P_{ac} + P_{ad} + P_{bc} + P_{bd} \geq (2 + 4\varepsilon)2^h$, then $f_{ac|bd} + f_{ad|bc} \leq 2K_\varepsilon\, 2^{h/2}$.

(d) If $P_{ab} + f_{ac|bd} + f_{ad|bc} \geq (1 + 2\varepsilon)2^h$, then $P_{cd} \leq 2K_\varepsilon\, 2^{h/2}$.

Proof. (a) is Theorem 5.5. In each of the other cases we change N into a pole N' (Figure 9) and apply Theorem 5.5 to two ports of N'.

(b) Add a new vertex z and join it to a and b. Then N' has rank $h + 1$, and $P_{zc}(N') = P_{ac} + P_{bc} \geq (\frac{1}{2} + \varepsilon)2^{h+1}$. The cycles of N' through z correspond to the paths of N from a to b. So $P_{ab} \leq c(N') \leq K_\varepsilon\, 2^{(h+1)/2}$.

(c) Add a new vertex z joined to a and b, and a new vertex w joined to c and d. Then N' has

rank $h+2$, and $P_{zw}(N')$ is the sum of the four path counts. The cycles of N' through z and w correspond to the pairs counted by $f_{ac|bd} + f_{ad|bc}$. So this sum is at most $K_\varepsilon\, 2^{(h+2)/2}$.

(d) Add an edge between c and d. Then N' has rank $h+1$. A path from a to b in N' avoids the new edge, or it consists of the new edge and a pair counted by $f_{ac|bd} + f_{ad|bc}$. So $P_{ab}(N') = P_{ab} + f_{ac|bd} + f_{ad|bc}$. The cycles of N' through the new edge correspond to the paths of N from c to d. So $P_{cd} \le K_\varepsilon\, 2^{(h+1)/2}$. ∎

6 Rings, small cyclic cuts, and cycles of length four

In this section G is a connected cubic multigraph of rank r.

IDEA | the results of this section

We count the cycles of the rings. Then we show that a graph of large rank with a cyclic cut of size at most three has at most $2^{r-1} + O(2^{r/2})$ cycles (Proposition 6.3), and that a graph without such a cut and with a cycle of length four is a ring, or it has fewer than 2^{r-1} cycles (Theorem 6.6). For the second statement we delete the cycle of length four and close the rest by a *cap*, a small tree (Lemma 6.4). The proof uses the gap of Section 5: a pole has all possible paths between two ends, or a fixed proportion of them is missing.

Let X be a set of $n \ge 1$ vertices of G with $|\delta X| = d$. Counting edge ends gives $\beta(G[X]) = (n-d)/2 + \mathrm{comp}(G[X])$. So $G[X]$ contains a cycle if $d \le 2$, and also if $d = 3$ and $n \ge 2$. Hence every cut of size at most two is cyclic, and a cut of size three is cyclic unless one of its sides is a single vertex. If G has a loop at a vertex v and $r \ge 2$, the other edge at v is a cyclic cut of size one.

6.1 Rings

A *ring* consists of pairwise disjoint tiles $\tau_1, \dots, \tau_m$ and of two disjoint edges from the right pair of τ_i to the left pair of τ_{i+1} for every i, with indices modulo m. These pairs of edges are the *interfaces*. Let the ring have a squares and s single rungs, and let $W = 2a + s$ be its weight. We assume that $W \ge 7$. Then the ring is a cubic graph with $2W$ vertices and rank $W+1$. Put

$$E(a,s) = 2^a\left(2a + 4s + \tbinom{s}{2} + \tfrac{1}{2}\right) - 3a - 4s + \tbinom{s}{2}.$$

Then $2^{2q} + E(q,0)$ and $2^{2q+1} + E(q,1)$ are the numbers $H(2q+1)$ and $H(2q+2)$ of (1.1).

Proposition 6.1 (Rings). A ring with $a \ge 1$ squares and s single rungs, of weight $W \ge 7$, has at most $2^W + E(a,s)$ cycles. Equality holds if $s \le 1$. A ring of weight $W \ge 7$ without squares has at most $2^W + W^2$ cycles.

Proof. Two interfaces together form a cut. So a cycle uses an odd number of edges of every interface, or an even number of edges of every interface.

One edge of every interface. There are 2^m ways to choose the edges. The cycle then enters each

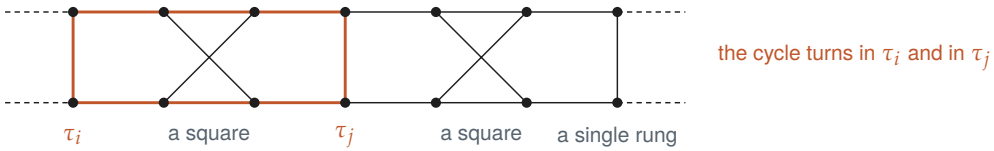


Figure 10. A part of a ring, and a cycle that uses both edges of every interface from τ_i to τ_j and no other interface. It turns in the single rungs τ_i and τ_j, and it passes the square between them on two disjoint edges.

tile at one vertex of its left pair and leaves it at one vertex of its right pair. In a square there are two such paths. In a single rung there is one: the cycle stays at the vertex or crosses the rung. Every choice gives a cycle. So there are $2^{m+a} = 2^W$ cycles of this kind.

No interface. These cycles are the a squares.

Both edges of every interface. In a square the cycle then uses two disjoint edges, and there are two ways to choose them. In a single rung it does not use the rung. Each of the 2^a choices is a cycle or a union of two cycles. Changing the choice in one square changes one into the other. So there are 2^{a-1} such cycles if $a \geq 1$, and at most one if $a = 0$.

Both edges of some interfaces, but not of all. Let the cycle use the interface after τ_i and not the interface before τ_i (Figure 10). Then it turns in τ_i: it joins the two vertices of the right pair inside the tile. It passes the next tiles on two disjoint paths, until it turns in a tile τ_j. This is already a closed curve, so the cycle uses exactly the interfaces from τ_i forward to τ_j. In each tile from τ_i to τ_j it has two possibilities if the tile is a square, and one if it is a single rung, in either role. Every choice gives a cycle. Let $I(i,j)$ be the set of the other tiles, those strictly after τ_j and strictly before τ_i. Then the number of these cycles is $2^a\,\omega(i,j)$, where $\omega(i,j) = 2^{-n}$ and n is the number of squares in $I(i,j)$. Every cycle of the present kind arises for exactly one ordered pair (i,j) of distinct tiles.

So a ring with $a \geq 1$ has $2^W + a + 2^{a-1} + 2^a\,\Omega$ cycles, where Ω is the sum of $\omega(i,j)$ over the ordered pairs of distinct tiles. We compute Ω. Let τ_j be a square. The squares $\tau_i \neq \tau_j$ give $\omega = 1, \frac{1}{2}, \dots, 2^{2-a}$, in the order in which they follow τ_j. So the pairs of two squares contribute $a(2 - 2^{2-a})$. Let τ_i be a single rung. The squares τ_j give $\omega = 1, \frac{1}{2}, \dots, 2^{1-a}$, in the order in which they precede τ_i. The same holds with the roles of i and j exchanged. So the pairs of a single rung and a square contribute $2s(2 - 2^{1-a})$. Finally let τ_i, τ_j be two single rungs, with n squares on one side between them and $a - n$ on the other. The two ordered pairs contribute $2^{-n} + 2^{-(a-n)} \leq 1 + 2^{-a}$, with equality if and only if $n = 0$ or $n = a$. Hence

$$\Omega \;\leq\; 2a + 4s + \tbinom{s}{2} - 2^{-a}\left(4a + 4s - \tbinom{s}{2}\right),$$

with equality if $s \leq 1$, because then there is no pair of two single rungs. This is the first statement.

If $a = 0$, every ordered pair of distinct tiles gives one cycle. There are $W(W-1)$ pairs, and at most one further cycle uses both edges of all interfaces. ■

Corollary 6.2 (Rings of a given rank)**.** Let $r \geq 8$. Every ring of rank r has at most $H(r)$ cycles. A ring of rank r with $\lfloor (r-1)/2 \rfloor$ squares and at most one single rung has exactly $H(r)$ cycles.

Proof. Let $W = r - 1$. A ring of weight W with $a = \lfloor W/2 \rfloor$ squares has $s = W - 2a \leq 1$ single rungs, and it has $2^W + E(a, s) = H(r)$ cycles by Proposition 6.1. For $a \geq 2$ and $2a + s = W$, a direct computation gives

$$E(a-1, s+2) - E(a, s) = -2^{a-1}\Big(W + \frac{s^2+s}{2} - \frac{13}{2}\Big) + 2s - 4 \;<\; 0,$$

because $W \geq 7$. So $2^W + E(a, s) \leq H(r)$ for every ring of weight W with a square. The number $H(r) - 2^W$ is 71 for $W = 7$ and 124 for $W = 8$, and it more than doubles when W grows by two. So it exceeds W^2, and a ring without squares has fewer than $H(r)$ cycles. ■

We write R_r for a ring of rank $r \geq 8$ with $\lfloor (r-1)/2 \rfloor$ squares and at most one single rung. In the notation of Horak and Kim [5] it is the graph MC_{r-1}, and they computed $c(R_r) = H(r)$.

6.2 Cuts of size at most three

In this subsection we use Theorem 5.5 and Lemma 5.9 with one number ε_0, and we put

$$\varepsilon_0 = \tfrac{1}{1000}, \qquad K_0 = K_{\varepsilon_0}, \qquad \kappa(h) = 2K_0\, 2^{-h/2}.$$

Proposition 6.3 (Cuts of size at most three)**.** There is a constant r_3 with the following property. Let G be a connected cubic multigraph of rank $r \geq r_3$ that has a cyclic cut of size at most three. Then

$$c(G) \;\leq\; 2^{r-1} + 5K_0^2\, 2^{r/2}.$$

Proof. Let $d \leq 3$ be the least size of a cyclic cut, and let δX be a cyclic cut of size d. By the remark at the start of this section, G has no cut of size less than d. So $G[X]$ and $G - X$ are connected, and no vertex is on two edges of δX if $d \geq 2$: otherwise we could move that vertex to the other side and get a smaller cut. Let T_1 and T_2 be the two sides, with ranks $b_1 \leq b_2$. Then $b_1 \geq 1$ and $r = b_1 + b_2 + d - 1$.

If $d = 1$, then $c(G) = c(T_1) + c(T_2) \leq (2^{b_1} - 1) + (2^{b_2} - 1) \leq 2^{r-1}$, because $b_1 + b_2 = r$.

Let $d \in \{2, 3\}$. Then G has no loop, so T_1 and T_2 are d-poles, and the cut edges match their ports. A cycle that crosses the cut uses exactly two cut edges, and it consists of these and of one path in each side. For a pair j of cut edges let $x_j\, 2^{b_1}$ and $a_j\, 2^{b_2}$ be the numbers of paths between the corresponding ports of T_1 and of T_2. Then $0 \leq x_j, a_j \leq 1$ by (5.1), and

$$c(G) \;=\; c(T_1) + c(T_2) + 2^{b_1+b_2} \sum_j x_j a_j, \qquad 2^{r-1} = 2^{d-2}\, 2^{b_1+b_2}. \tag{6.1}$$

There is one pair for $d = 2$, and there are three pairs for $d = 3$. Let r be large. Then $b_2 \geq (r-2)/2$ is large, and $\kappa(b_2) \leq \frac{1}{1000}$. We have $c(T_1) < 2^{b_1}$.

Case A: $a_j \le 0.501$ **for all** j. By Corollary 3.3, $c(T_2) \le \frac{9}{16}2^{b_2} \le \frac{9}{32}2^{b_1+b_2}$. So $c(G)\,2^{-b_1-b_2}$ is less than $2^{-b_2} + \frac{9}{32} + 0.501$ for $d = 2$, and less than $2^{-b_2} + \frac{9}{32} + 1.503$ for $d = 3$. In both cases $c(G) < 2^{r-1}$.

Case B: $a_j > 0.501$ **for some** j. Then $c(T_2) \le K_0\,2^{b_2/2}$ by Lemma 5.9(a). So $c(T_1) + c(T_2) < 2^{b_1} + K_0\,2^{b_2/2} \le \frac{1}{1000}2^{b_1+b_2}$.

Let $d = 2$. If $x_1 < 0.501$, then $c(G) < 0.502 \cdot 2^{b_1+b_2} < 2^{r-1}$ by (6.1). Otherwise $c(T_1) \le K_0\,2^{b_1/2}$ by Lemma 5.9(a), and

$$c(G) \;\le\; 2^{r-1} + K_0\big(2^{b_1/2} + 2^{b_2/2}\big) \;\le\; 2^{r-1} + 2K_0\,2^{r/2}.$$

Let $d = 3$, and number the pairs so that $a_1 \ge a_2 \ge a_3$. Any two pairs of ports of a 3-pole have one port in common. If $a_1 + a_2 < 1.002$, then $\sum_j x_j a_j < 1.503$ and $c(G) < 2^{r-1}$. Otherwise $a_3 \le \kappa(b_2)$ by Lemma 5.9(b). If $x_1 + x_2 < 1.002$, then $\sum_j x_j a_j < 1.003$ and $c(G) < 2^{r-1}$. Otherwise $x_3 \le \kappa(b_1)$ and $c(T_1) \le K_0\,2^{b_1/2}$ by Lemma 5.9(b) and (a). Then (6.1) gives

$$c(G) \;\le\; 2K_0\,2^{b_2/2} + 2^{b_1+b_2}\big(2 + \kappa(b_1)\kappa(b_2)\big) \;=\; 2^{r-1} + 2K_0\,2^{b_2/2} + 4K_0^2\,2^{(b_1+b_2)/2} \;\le\; 2^{r-1} + 5K_0^2\,2^{r/2}.$$

■

6.3 A pole with four ports

Let Y be a 4-pole of rank h with ports $1, 2, 3, 4$. A *pairing* is one of the three partitions $12|34$, $13|24$, $14|23$ of the ports into two pairs; the two pairs of a pairing are *opposite*. For a pair ij and a pairing $\alpha = ij|kl$ put

$$a_{ij} = 2^{-h}P_{ij}(Y), \qquad u_\alpha = 2^{-h}f_{ij|kl}(Y), \qquad S = \sum_{ij} a_{ij}, \qquad F = \sum_{\alpha} u_\alpha.$$

A pair of disjoint paths that joins the four ports is a join for the set of all four ports, and there are 2^h such joins. So $a_{ij} \le 1$ and $F \le 1$.



the graph G the cap for $14|23$

Figure 11. A 4-pole Y. Left: the graph G of Section 6.4, in which a cycle C of length four is joined to the four ports. Right: the cap of Lemma 6.4 for the pairing $14|23$. The number a_{ij} counts the paths between the ports i and j of Y, and u_α counts the pairs of disjoint paths for a pairing α of the ports, both as a proportion of the largest possible number.

Lemma 6.4 (Cap). For every $\eta > 0$ there is $h_1(\eta)$ with the following property. Let Y be a 4-pole of rank $h \geq h_1(\eta)$, and put $z = 2^{-h}c(Y)$. Then, for every pairing α,

$$z + S + F - u_\alpha \;\leq\; 4 + \eta.$$

Proof. Let $\alpha = ij|kl$. Add two new vertices x and y and an edge between them, join x to the ports i and j, and join y to the ports k and l (Figure 11). The new multigraph $\hat{Y}$ has two more vertices and five more edges than Y, so its rank is $h + 3$. The new edges form a tree, and this tree joins any two ports by a path. A path of Y between two ports and the path of the tree between its ends form a cycle. A pair of disjoint paths of Y whose pairing is not α and the four edges of the tree at the ports form a cycle. With the cycles of Y, these are $c(Y)+(S+F-u_\alpha)2^h$ distinct cycles of $\hat{Y}$. By Corollary 3.3, $\hat{Y}$ has at most $(\frac{1}{2}+\frac{1}{8}\eta)\,2^{h+3} = (4+\eta)\,2^h$ cycles, when h is large. ∎

By Lemma 6.4, the number $S + F - u_\alpha$ is at most $4 + o(1)$ for every pairing α. The next lemma describes the poles in which $S + F$ is at least $4 - o(1)$. They have one of two shapes (Figure 12).



three pairs with a common port — the four pairs between two pairs of ports

Figure 12. The two shapes of Lemma 6.5. A line joins two ports whose pair is full.

Lemma 6.5 (Rigidity). There are numbers $\rho_0 \in (0, 1)$ and h_0 with the following property. Let Y be a 4-pole of rank $h \geq h_0$ with $S + F \geq 4 - \rho_0$. Then three pairs with a common port are full, or the four pairs between two disjoint pairs of ports are full.

Proof. Suppose that this is false. Then there are 4-poles $Y_1, Y_2, \ldots$ whose ranks h_n tend to infinity, with $S(Y_n) + F(Y_n) \geq 4 - \frac{1}{n}$, such that no Y_n has three full pairs with a common port or four full pairs between two disjoint pairs of ports. The numbers $a_{ij}(Y_n)$ and $u_\alpha(Y_n)$ lie in $[0, 1]$. We take a subsequence in which each of these nine sequences converges, and we write a_{ij}, u_α, S and F for the limits. Then $F \leq 1$ and $S + F \geq 4$.

Three implications. For distinct ports i, j, k, l,

$$a_{ik} + a_{jk} > 1 \implies a_{ij} = 0; \tag{6.2}$$

$$a_{ik} + a_{il} + a_{jk} + a_{jl} > 2 \implies u_{ik|jl} + u_{il|jk} = 0; \tag{6.3}$$

$$a_{kl} > 0 \implies a_{ij} + F - u_{ij|kl} \leq 1. \tag{6.4}$$

Indeed, let $a_{ik}+a_{jk} > 1+2\varepsilon$ with $\varepsilon > 0$. For large n, Lemma 5.9(b) gives $a_{ij}(Y_n) \leq 2K_\varepsilon\, 2^{-h_n/2}$, and this tends to 0. In the same way (6.3) follows from Lemma 5.9(c). If $a_{ij} + F - u_{ij|kl} > 1$, then Lemma 5.9(d) gives $a_{kl} = 0$; this is (6.4).

The limit. For a partition of the ports into two pairs, the *cross sum* is the sum of a_{ij} over the four pairs ij that are not pairs of the partition. Each pair occurs in two of the three cross sums.

Suppose that some cross sum is larger than 2, say the cross sum X of $12|34$. Then $a_{13}+a_{23} > 1$ or $a_{14}+a_{24} > 1$, so $a_{12} = 0$ by (6.2). In the same way $a_{34} = 0$. By (6.3), $u_{13|24} = u_{14|23} = 0$. Put $u = u_{12|34}$. Then $X + u = S + F \geq 4$. If one of the four numbers in X is 0, say a_{24}, then the other three and u are equal to 1. This contradicts (6.4) for $ij = 14$, because $a_{23} > 0$. So the four numbers are positive, and (6.4) gives $a_{ij} + u \leq 1$ for each of them. Hence $X + 4u \leq 4$. With $X + u \geq 4$ this gives $u = 0$ and $X = 4$, so the four numbers are equal to 1.

Suppose that no cross sum is larger than 2. The three cross sums add up to $2S$, so $S \leq 3$. Hence $S = 3$ and $F = 1$, every cross sum is 2, and two opposite numbers add up to 1. If all six numbers are positive, then (6.4) gives $u_\alpha \geq \max\{a_{ij}, a_{kl}\} \geq \frac{1}{2}$ for every pairing $\alpha = ij|kl$, and this contradicts $F = 1$. So one number is 0, say a_{34}, and $a_{12} = 1$. If $a_{13} > 0$, then $a_{23} = 0$ by (6.2), so $a_{14} = 1$; then $a_{24} = 0$ by (6.2), so $a_{13} = 1$. If $a_{13} = 0$, then $a_{24} = 1$; then $a_{14} = 0$ by (6.2), so $a_{23} = 1$. In both cases three numbers with a common port are equal to 1.

The end. So three numbers a_{ij} with a common port, or the four numbers between two disjoint pairs of ports, are equal to 1. For large n the corresponding numbers $a_{ij}(Y_n)$ are larger than $1 - \gamma_0$, and these pairs are full in Y_n by Theorem 5.7. This is a contradiction. ■

6.4 Cycles of length four

IDEA | outline of the proof of Theorem 6.6

Let C be a cycle of length four, and let Y be the rest of the graph, of rank k. A cycle that meets C and Y consists of paths of Y and of paths of C. In units of 2^k, the number of these cycles is the sum of the numbers $S + F - u_\alpha$ for the two pairings α that pair the ports at neighbours on C. By the cap of Lemma 6.4, one of these numbers and the proportion of cycles of Y are together at most $4 + o(1)$, and $c(G) \geq 2^{r-1}$ requires the sum 8. So $S + F \geq 4 - o(1)$. By Lemma 6.5 there are two possible shapes. Three full pairs at one port give a cyclic cut of size three. If the four pairs between two pairs of ports are full, then Y is a chain of tiles, and G is a ring.

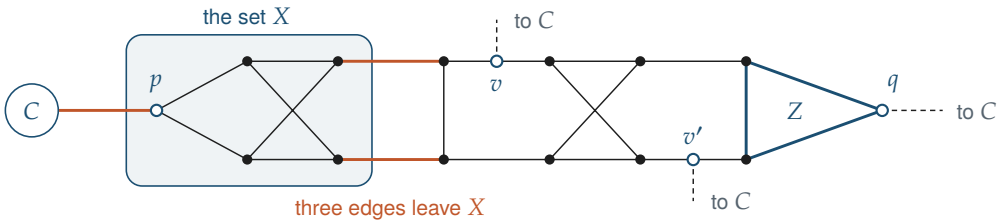


Figure 13. Case 1 in the proof of Theorem 6.6. Without the ports v and v', the pole Y is a capped ladder with the caps p and q. The ports v and v' are not on an edge inside the set X, because a path from p to such a port avoids the cycle Z.

Theorem 6.6 (A cycle of length four)**.** There is a constant r_4 with the following property. Let G be a connected cubic multigraph of rank $r \geq r_4$ that has no cyclic cut of size at most three and has a cycle of length four. Then G is a ring, or $c(G) < 2^{r-1}$.

Proof. Let $C = c_1c_2c_3c_4$ be a cycle of length four, and put $Y = G - V(C)$ (Figure 11). By the remark at the start of the section, G has no loop and no cut of size at most two, and every cut of size three has a single vertex on one side. The graph has $2r - 2$ vertices, so a set of at most five vertices and its complement both have more than one vertex.

The rest is a 4-pole without bridge. No edge outside C joins two vertices of C: such an edge and a path of C form a cycle with at most three vertices, and the cut around these vertices has size at most three. So exactly four edges leave $V(C)$, one at each vertex. Let c_i be joined to the vertex i of Y. The vertices $1, 2, 3, 4$ are distinct: if two of them are one vertex v, the cut around $V(C) \cup \{v\}$ has size at most three. The multigraph Y is connected: otherwise some component of Y is joined to C by at most two edges, and these form a cut. So Y is a 4-pole with the ports $1, 2, 3, 4$. Let k be its rank. Then $r = k + 4$.

Let e be a bridge of Y, and let A and B be the vertex sets of the two components of $Y - e$. In $G[A]$ every vertex has degree at least two, except the end of e, which has degree at least one. A forest with an edge has two vertices of degree one, so $G[A]$ contains a cycle. The same holds for $G[B]$. Each of the four edges between C and Y is in exactly one of the cuts δA and δB, and e is in both. So $|\delta A| + |\delta B| = 6$, and one of them is a cyclic cut of size at most three. Hence Y has no bridge.

The count. We use the notation of this section for Y, with $h = k$, and we put $z = 2^{-k}c(Y)$. A cycle of G that is neither C nor a cycle of Y uses two or four of the edges between C and Y. With two, at c_i and c_j, it consists of a path of Y between the ports i and j and of one of the two paths of C between c_i and c_j. With four, it consists of a pair of disjoint paths of Y and of two disjoint edges of C. The edges c_1c_2 and c_3c_4 close the pair to one cycle if and only if its pairing is not $12|34$. The edges c_1c_4 and c_2c_3 do this if and only if its pairing is not $14|23$. Hence

$$\frac{c(G)}{2^k} \;=\; 2^{-k} + z + \big(S + F - u_{12|34}\big) + \big(S + F - u_{14|23}\big). \tag{6.5}$$

Let ρ_0 and h_0 be as in Lemma 6.5, and let r be so large that $k \geq \max\{h_0,\, h_1(\rho_0/2),\, 4\}$ and $2^{-k} \leq \rho_0/2$. Suppose that $c(G) \geq 2^{r-1} = 8 \cdot 2^k$. We show that G is a ring. By Lemma 6.4 for the pairing $14|23$, the term z and the second bracket of (6.5) are together at most $4 + \rho_0/2$. So $S + F \geq S + F - u_{12|34} \geq 4 - \rho_0$, and Lemma 6.5 applies to Y. It gives two cases.

Case 1: three pairs with a common port are full. Let p be the common port, and let q, v and v' be the other ports. Replace v and v', each with its two edges, by one edge. By Lemma 5.2(a) the result Y° is a 2-pole of rank k with the ports p, q, and this pair is full. It has no bridge, because Y has none. By Lemma 5.6, Y° is a capped ladder with the caps p and q. Its tiles have total weight $k - 1 \geq 3$, so there are at least two tiles. Let X consist of p and of the vertices of the first tile (Figure 13), and let Z be a cycle of Y° in the last tile and the cap q. Then Z avoids X, and $Y^\circ[X]$ is connected and contains a cycle.

Suppose that the edge of Y° that replaces v has both ends in X. Then Y has a path from p to v whose other vertices are in $X \cup \{v'\}$. It is disjoint from the cycle of Y that corresponds to Z. This contradicts Lemma 5.1 for the full pair p, v. The same holds for v'. So the edges of Y° with both ends in X are edges of Y, and $G[X]$ contains a cycle. Three edges of G leave X: the edge from p to C, and one edge of Y for each of the two edges of Y° that leave X. The cycle C is outside X. So δX is a cyclic cut of size three, and this case does not occur.

Case 2: the four pairs between two disjoint pairs of ports are full. The pole Y has no bridge. By Corollary 5.8 it is a chain of tiles; one of the two pairs of ports is the left pair of its first tile, and the other is the right pair of its last tile. If the pairs are $1, 3$ and $2, 4$, then C is a square with the pairs c_1, c_3 and c_2, c_4. In the other case let the pairs be $1, 2$ and $3, 4$, by symmetry. Then the edges c_1c_2 and c_3c_4 are single rungs, and the edges c_2c_3 and c_1c_4 are the interface between them. In both cases the edges between C and Y are two interfaces. So G is a ring. ■

7 Girth five outside a small set

Recall that $q(G, P)$ is the probability that $\xi \cap E(G - P)$ is a forest.

Theorem 7.1 (Girth five). There is an integer r_5 with the following property. Let G be a connected cubic multigraph of rank $r \geq r_5$. Let P be a nonempty set of at most $3 \log r$ vertices such that $G[P]$ is connected. If $G - P$ has girth at least five, then $q(G, P) < \frac{2}{5}$.

IDEA | outline of the proof

Suppose that $q(G, P) \geq \frac{2}{5}$. We find two disjoint parts of $G - P$ that are large and have small boundaries. Each of them has a forest proportion of at most $\frac{3}{5}$, for every choice of the edges outside it. This is a contradiction, because $(\frac{3}{5})^2 < \frac{2}{5}$. For a part with many boundary vertices this needs that the part contains no short cycle (Section 7.1). For a part with few boundary vertices it needs girth five only, because a ladder has cycles of length four (Section 7.2). The parts come from a family of regions with a large total weight: the second moment gives many pairs with exactly two links, and these pairs give nested separators (Section 7.3). We use this for short cycles (Section 7.4) and for cycles of moderate length (Section 7.5).

7.1 Forest proportions of a piece

A *piece* is the subgraph of G that a connected set of vertices outside P induces. It is a connected subcubic multigraph, and we use the following notions for it.

NOTATION | joins and forest proportions

- N is a connected subcubic multigraph. Its vertices of degree less than three are its *boundary*

vertices, and its *deficit* $d(N)$ is the sum of $3 - \deg v$ over all vertices v. A count of edge ends gives

$$|V(N)| = 2\beta(N) - 2 + d(N). \tag{7.1}$$

- T is a set of boundary vertices of even size. A *T-join* of N is a set of edges in which the vertices of T have odd degree and all other vertices have even degree. There are $2^{\beta(N)}$ of them.
- The *forest proportion* of T is the proportion of the T-joins that are forests, and T is *rigid* if every T-join is a forest. For $T = \emptyset$ the T-joins are the even subgraphs, only the empty one is a forest, and the forest proportion is $2^{-\beta(N)}$.
- $R_* = R_{1/20}$ and $B = 2R_* + 2^{100} + 1$, with $R_{1/20}$ as in Section 4. Then $\Theta_{1/20}(k) + k \le B(k+1)^2$ for all $k \ge 1$.

When the edges of an even subgraph of G outside a piece are given, its edges in the piece form a T-join for some set T. So a bound for all forest proportions of a piece gives a factor in the estimate of $q(G, P)$, and the factors of disjoint pieces multiply (Lemma 7.3). The first lemma is for a piece of large girth.

Lemma 7.2 (Forest proportions at large girth)**.** There are absolute constants $C_0 \ge 1$ and $C_1 \ge 1$ with the following property. Let $\Lambda \ge 4$, and let N be a connected subcubic multigraph with at most 2Λ boundary vertices, at least $C_1\Lambda^4$ vertices, and girth at least $C_0 \log(\Lambda + 2)$. Then every set of boundary vertices of even size has forest proportion less than $\frac{3}{5}$.

Proof. Let s be the least integer with $2^s > (2s+1)B(2\Lambda+1)^2$. Then $2^s \le 2(2s+1)B(2\Lambda+1)^2$, and B is an absolute constant. So there are absolute constants $C_0 \ge 1$ and $C_1 \ge 1$ such that $2s < C_0 \log(\Lambda + 2)$ and $6\Lambda\, 2^s < C_1\Lambda^4$ for every $\Lambda \ge 4$.

Let k be the number of boundary vertices of N. If $k \le 1$, the only set is the empty one. Then N contains a cycle, and the forest proportion is at most $\frac{1}{2}$. Let $k \ge 2$, fix a boundary vertex o, and call the set of the vertices at distance j from o the *layer* j.

A large layer. A ball of radius s has at most $3 \cdot 2^s$ vertices. So at most $3k\, 2^s \le 6\Lambda\, 2^s$ vertices are within distance s of a boundary vertex, and some vertex w has distance more than s from every boundary vertex. All vertices within distance s of w have degree three, and $2s$ is less than the girth. So the ball of radius s around w is a tree in which every vertex at distance less than s from w has degree three, and it has at least 2^s vertices. The distances from o to two vertices of this ball differ by at most $2s$. So the ball meets at most $2s+1$ layers, and some layer has at least $2^s/(2s+1) > B(k+1)^2$ vertices.

A marker. Suppose that some set T of boundary vertices has forest proportion at least $\frac{3}{5}$. Take a breadth-first tree of N from o, and let P be the union of its paths from o to the other boundary vertices. Then $N[P]$ is connected, every vertex outside P has degree three, and P has at most k vertices in each layer, because a path of the tree from o has one vertex in each layer that it meets. The tree paths inside P contain a T-join F_0. The map $J \mapsto J + F_0$ is a bijection from the T-joins of N to the even subgraphs of N, and $J + F_0$ has the same edges in $N - P$ as J. So if J is a forest, then $J + F_0$ is a forest in $N - P$. Hence $q(N, P) \ge \frac{3}{5} > \frac{1}{2} + \frac{1}{20}$.

The distance from o is a level function. By Lemma 4.1 with $\delta = \frac{1}{20}$, each layer has fewer than $\Theta_{1/20}(k)$ vertices outside P, and it has at most k vertices in P. So each layer has fewer than $B(k+1)^2$ vertices. This is a contradiction. ■

Lemma 7.3 (Product). Let N be a multigraph, and let P and T be sets of vertices such that N has a T-join. Let $U_1, \dots, U_m \subseteq V(N) \setminus (P \cup T)$ be pairwise disjoint sets of vertices of degree three such that each $N[U_i]$ is connected. Let $f_1, \dots, f_m$ be numbers such that in $N[U_i]$ every even set of boundary vertices has forest proportion at most f_i. Then $J \cap E(N - P)$ is a forest for at most the proportion $f_1 \cdots f_m$ of the T-joins J of N. For a cubic multigraph G and $T = \emptyset$ this says that $q(G, P) \le f_1 \cdots f_m$.

Proof. Let E' consist of the edges that lie in none of the graphs $N[U_i]$, and let J be a uniformly random T-join. Fix a set $x \subseteq E'$ that extends to a T-join. For each i, let T_i be the set of the vertices of U_i that are incident with an odd number of edges of x. The set U_i avoids T, so the parity equations say that $J \cap E(N[U_i])$ is a T_i-join. These equations involve disjoint sets of edges, so the joins are independent and uniform, conditional on $J \cap E' = x$. A vertex of degree three in $N[U_i]$ has no incident edge in E'. So T_i is an even set of boundary vertices. If $J \cap E(N - P)$ is a forest, its restriction to every $N[U_i]$ is a forest. The conditional probability is therefore at most $f_1 \cdots f_m$. Average over x. ■

7.2 Pieces with few boundary vertices at girth five

IDEA | outline of Lemmas 7.4 and 7.5

Lemma 7.2 needs a girth that grows with the number of boundary vertices. For a bounded number of boundary vertices, girth five is sufficient. Girth four is not sufficient: in a ladder, the two ends of one rail form a rigid set. The proof has two steps.

1. A multigraph of girth five with a full pair is small. It consists of capped ladders, and a capped ladder has many cycles with at most four edges (Lemma 7.4).
2. A large piece with a forest proportion of at least $\frac{3}{5}$ is thin and long. So it contains many disjoint parts that are thin and long. Such a part has many disjoint slabs. For one of the slabs, a rigid set of the part would give a full pair in the slab. So the part has no rigid set, and each part loses a fixed proportion of the forests (Lemma 7.5, Figure 14).

Lemma 7.4 (Full pairs at girth five). Let M be a connected subcubic multigraph of girth at least five. Let $x \neq y$ be vertices of degree at most two such that every $\{x, y\}$-join of M is a path. Then $|V(M)| < 15\, d(M)$.

Proof. Put $d = d(M)$ and $b = \beta(M)$. If $b = 0$, then M is a tree and $|V(M)| = d - 2$. Let $b \ge 1$. Delete a vertex of degree one other than x and y, as long as there is one. If then x

has degree one, delete x and take its neighbour in the place of x; do the same for y; repeat. These steps keep the rank, and every join is a path after each step. The two vertices stay distinct, because $b \geq 1$. The result M_1 is a pole in which the two vertices are ports and form a full pair. A port of M_1 has degree two in M, or one of its edges in M leads to a deleted tree, and such a tree has a deficit of at least two. So M_1 has at most d ports.

By Lemma 5.2, replace the other ports by edges. The result M_1° consists of bridges and of bridgeless 2-poles $N_1, \dots, N_q$ with full pairs of ports, and their ranks $b_i \geq 1$ add up to b. An edge of M_1° is a path of M_1 whose inner vertices are ports; call it *plain* if this path has no inner vertex. At most $d - 2$ edges are not plain. A cycle of M_1° with at most four edges is not a cycle of M, so one of its edges is not plain. By Lemma 5.6, N_i is a pair of parallel edges, or a capped ladder with some number m of tiles. In the second case its rank is at most $2m + 1$, and it contains at least $m/3$ pairwise edge-disjoint cycles with at most four edges: the squares, pairs of consecutive single rungs with the interface between them, and one cycle through a cap. So in both cases at least $b_i/7$ edges of N_i are not plain. Hence $b \leq 7(d - 2)$, and (7.1) gives $|V(M)| < 15d$. ■

Lemma 7.5 (Pieces at girth five). For every integer $k \geq 1$ there is a number $F_0(k)$ with the following property. Let N be a connected subcubic multigraph of girth at least five with at most k boundary vertices and at least $F_0(k)$ vertices. Then every set of boundary vertices of even size has forest proportion less than $\frac{3}{5}$.

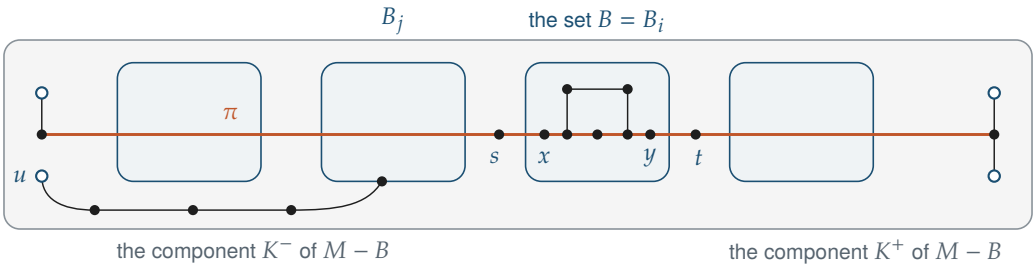


Figure 14. The proof of Lemma 7.5. The part M with the path π (thick), four of its slabs, and the vertices of U (circles) in the first and in the last layer. The vertex u is in a side component of $M - B_j$; for the slab B it is in K^-. If $K^- \neq K^+$ and each of them contains an odd number of vertices of U, then a U-join of M meets $M[B]$ in an $\{x, y\}$-join.

Proof. Put $W = B(k+1)^2$, $\tau = 60\,W$, $h = 2W\tau + 2$ and $F_0(k) = Wh\,(k + 2^{Wh} + 2)$. Suppose that some even set T of boundary vertices has forest proportion at least $\frac{3}{5}$. If T is empty, then N has no cycle and fewer than $2k$ vertices. In the other case fix a boundary vertex o, and define the layers as in the proof of Lemma 7.2. The step "A marker" of that proof uses no girth, and it shows that each layer has fewer than W vertices. So N has at least $h\,(k + 2^{Wh} + 2)$ layers. Divide the layers $1, 2, \dots$ into groups of h consecutive layers. At least 2^{Wh} groups contain no boundary vertex. In such a group let M be a component of the subgraph induced by the group that has a vertex v in the last layer of the group. A shortest path from o to v has

one vertex in each layer of the group; let π_j be its vertex in the jth of these layers. The path $\pi = \pi_1 \cdots \pi_h$ lies in M. The vertices of M have degree three in N, and an edge that leaves M starts in the first or in the last layer of the group. So the boundary vertices of M are in these two layers, there are fewer than $2W$ of them, and the rank of M is less than Wh. We show that no even set of boundary vertices of M is rigid (Figure 14).

Slabs. For $1 \le i \le 2W$, the ith *slab* consists of the layers $2+(i-1)\tau, \dots, 1+i\tau$ of the group. Let B_i be the vertex set of the component of the subgraph of M induced by this slab that contains the τ vertices of π in the slab. Let $s_i x_i$ be the edge of π that enters B_i and $y_i t_i$ the edge that leaves it, with $x_i, y_i \in B_i$. The vertices of B_i have degree three in M, and an edge of M that leaves B_i starts in the first or in the last layer of the slab. So $d(M[B_i]) < 4W \le |B_i|$. Hence $M[B_i]$ has a cycle, by (7.1), and $x_i \ne y_i$ are boundary vertices of $M[B_i]$. Call a component of $M - B_i$ a *side component* if it contains neither s_i nor t_i. The vertices of π outside B_i are joined to s_i or to t_i in $\pi - B_i$. A set B_j with $j \ne i$ is connected, avoids B_i and contains a vertex of π. So a side component of $M - B_i$ meets no set B_j.

Side components. Let K be a side component of $M - B_i$, and let $j \ne i$. Some vertex of K has a neighbour in B_i, because M is connected. So $K \cup B_i$ is connected, avoids B_j and contains a vertex of π. Hence it lies in the component of s_j or of t_j in $M - B_j$. So a vertex of M is in a side component of $M - B_i$ for at most one i.

No rigid set. Let U be an even set of boundary vertices of M. It has fewer than $2W$ vertices, and none of them is in a slab. So there is an i such that no vertex of U is in a side component of $M - B_i$. Write B, s, x, y, t for B_i, s_i, x_i, y_i, t_i, and let K^- and K^+ be the components of $M - B$ that contain s and t. Every vertex of U is in one of them. Put $U^- = U \cap V(K^-)$. Suppose first that $K^- = K^+$ or that $|U^-|$ is even. Then $M - B$ contains a U-join, and its union with a cycle of $M[B]$ is a U-join that is not a forest. Suppose now that $K^- \ne K^+$ and that $|U^-|$ is odd, and let J be an $\{x, y\}$-join of $M[B]$. Add the edges sx and ty, a join of K^- for the set U^- with s added or removed, and a join of K^+ for the set $U \setminus U^-$ with t added or removed. The result is a U-join of M. If U is rigid, then J is a forest, and hence a path. Then Lemma 7.4 gives $|B| < 15\, d(M[B]) < 60\, W = \tau$, and this is false.

So each even set of boundary vertices of M has forest proportion at most $1 - 2^{-Wh}$. By Lemma 7.3, the forest proportion of T is at most $(1-2^{-Wh})^{2^{Wh}} < \frac{1}{2}$. This is a contradiction. ■

NOTATION | constants and short cycles for the rest of this section

- C_0 and C_1 are as in Lemma 7.2, and F_0 is as in Lemma 7.5.
- $\Lambda_0 \ge 7$ is a fixed integer such that every integer $d \ge \Lambda_0$ satisfies

$$\big(C_0 \log(d+2)\big)^{C_0+9} \le d, \qquad (d+2)^{C_0} \le 2\, d^{\,C_0}, \qquad d^2 \ge 20\, C_1.$$

  Then $C_0 \log(d+2) \le d$.
- $C_2 \ge 4$ is a number with $C_2 \ge F_0(2\Lambda)$ for $4 \le \Lambda < \Lambda_0$.
- For an integer $\Lambda \ge 4$ we put $\Phi(\Lambda) = C_1 \Lambda^4$ if $\Lambda \ge \Lambda_0$, and $\Phi(\Lambda) = C_2$ if $\Lambda < \Lambda_0$.
- A cycle is Λ-*short* if $\Lambda \ge \Lambda_0$ and its length is less than $C_0 \log(\Lambda + 2)$. So no cycle is Λ-short if $\Lambda < \Lambda_0$.

- For a multigraph G and a set P of vertices, $\nu_\Lambda(G, P)$ is the largest number of pairwise disjoint Λ-short cycles of $G - P$.

Corollary 7.6 (Pieces). Let $\Lambda \geq 4$ be an integer. Let N be a connected subcubic multigraph of girth at least five with at most 2Λ boundary vertices, at least $\Phi(\Lambda)$ vertices and no Λ-short cycle. Then every even set of boundary vertices of N has forest proportion less than $\frac{3}{5}$.

Proof. If $\Lambda \geq \Lambda_0$, then N has no cycle of length less than $C_0 \log(\Lambda + 2)$, and Lemma 7.2 applies. If $\Lambda < \Lambda_0$, then N has at least $C_2 \geq F_0(2\Lambda)$ vertices, and Lemma 7.5 applies with $k = 2\Lambda$. ■

7.3 Separators

IDEA | two links give a separator

If two disjoint connected sets have exactly two links, then one of them separates a neighbour of the other one from a fixed vertex p. Disjoint sets that separate the same two vertices are nested, and the parts between them have small boundaries (Figure 15). So a long chain of such sets contains two pieces without a short cycle.

Lemma 7.7 (Nested separators). Let u and p be two vertices of a connected graph H. Let C_1 and C_2 be disjoint connected sets of vertices that avoid u and p, such that u and p lie in different components of $H - C_i$ for $i = 1, 2$. Let U_i be the vertex set of the component of u in $H - C_i$. Then exactly one of the following holds: $C_1 \subseteq U_2$ and $U_1 \subseteq U_2$; or $C_2 \subseteq U_1$ and $U_2 \subseteq U_1$.

Proof. Let W_i be the vertex set of the component of p in $H - C_i$. Some edge joins U_i to C_i, and some edge joins W_i to C_i, because H is connected. The set C_2 is connected and disjoint from C_1, so it lies in one component of $H - C_1$. If this component is not U_1, then $U_1 \cup C_1$ is a connected set that contains u and avoids C_2, so $U_1 \cup C_1 \subseteq U_2$. This is the first alternative. If $C_2 \subseteq U_1$, then $W_1 \cup C_1$ is a connected set that contains p and avoids C_2, so $W_1 \cup C_1 \subseteq W_2$. Then $U_2 \cup C_2$ is a connected set that contains u and avoids C_1, so $U_2 \cup C_2 \subseteq U_1$. This is the second alternative. The two alternatives exclude each other, because $C_1 \subseteq U_2$ and $U_2 \subseteq U_1$ would give $C_1 \subseteq U_1$. ■

So a family of pairwise disjoint connected sets that separate u from p is linearly ordered: write $C_1 < C_2$ in the first case of the lemma. The order is transitive, because it is the order of the sets U_i by inclusion.

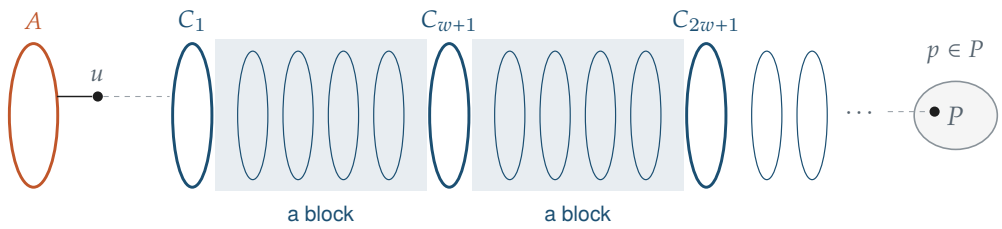


Figure 15. Nested separators. Each set C_i separates the vertex u from p in the graph without the region A. So the sets C_i are nested (Lemma 7.7). Every w-th set is the end of a block, and the edges that leave a block end in its two end sets (proof of Proposition 7.8).

Proposition 7.8 (A long chain)**.** Let G be a connected cubic multigraph, and let P be a nonempty set of vertices such that $G[P]$ is connected and $G - P$ has girth at least five. Let $p \in P$, let A be a region that avoids P, and let $u \notin A$ be a vertex in the component of p in $G - A$. Let $C_1, \dots, C_T$ be pairwise disjoint regions that avoid P, A and u and separate u from p in $G - A$. Let $\Lambda \geq 4$ be an integer and $m \geq 1$ a number such that A and every C_i have at most Λ boundary edges, and every C_i has at least m vertices. If

$$T \;>\; \bigl(\Lambda + \nu_\Lambda(G,P) + 2\bigr)\bigl(\lceil 2\Lambda\,\Phi(\Lambda)/m\rceil + 1\bigr),$$

then $q(G,P) \leq \frac{9}{25}$.

Proof. Each C_i is connected and separates u from p in $G - A$. So it lies in the component H of $G - A$ that contains u and p. By Lemma 7.7 we may assume that $C_1 < C_2 < \cdots < C_T$. Let U_i be the vertex set of the component of u in $H - C_i$. Then $U_i \cup C_i \subseteq U_j$ for $i < j$.

Blocks. For $i < j$ put $X_{ij} = U_j \setminus (U_i \cup C_i)$. An edge of H that leaves U_j ends in C_j, because U_j is a component of $H - C_j$. No edge joins X_{ij} to U_i: its end in X_{ij} would lie in the component U_i of $H - C_i$. So every edge of H that leaves X_{ij} ends in C_i or in C_j. If $i < l < j$, then $C_l \subseteq U_j$, and C_l avoids $U_i \cup C_i \subseteq U_l$; so $C_l \subseteq X_{ij}$. If $i < j \leq i' < j'$, then $X_{ij} \subseteq U_j \subseteq U_{i'}$ is disjoint from $X_{i'j'}$.

Put $w = \lceil 2\Lambda\,\Phi(\Lambda)/m\rceil + 1$, and let $Y_l = X_{lw+1,\,(l+1)w+1}$ for $l = 0, 1, \dots$ with $(l+1)w \leq T - 1$. These sets are pairwise disjoint, each contains $w - 1$ of the sets C_i, and every edge of H that leaves Y_l ends in one of its two end sets. So $|Y_l| \geq (w-1)m \geq 2\Lambda\,\Phi(\Lambda)$, and at most 2Λ edges of H leave Y_l.

There are at least $\Lambda + \nu + 2$ sets Y_l, where $\nu = \nu_\Lambda(G,P)$. The set P is connected, it contains p, and it is disjoint from A and from the sets C_i. So P lies in the component of p in $H - C_i$ for every i, and every Y_l avoids P. Discard Y_l if it contains a vertex with a neighbour in A, or if $G[Y_l]$ contains a Λ-short cycle. The sets Y_l are pairwise disjoint, so this discards at most $\Lambda + \nu$ sets, and two sets remain. Let Y be one of them. The edges of G that leave Y are edges of H, so $|\delta Y| \leq 2\Lambda$.

Pieces. Every component of $G[Y]$ has an edge that leaves Y. So there are at most 2Λ components, and one of them has a vertex set Y' with at least $\Phi(\Lambda)$ vertices. A boundary vertex of

$G[Y']$ is an end of an edge of δY, so there are at most 2Λ of them, and $G[Y']$ has girth at least five and no Λ-short cycle. So every even set of boundary vertices of $G[Y']$ has forest proportion less than $\frac{3}{5}$, by Corollary 7.6. Lemma 7.3, for the two sets Y', gives $q(G,P) \leq \frac{9}{25}$. ■

Lemma 7.9 (Two links)**.** Let G be a connected multigraph, and let p be a vertex. Let $\mathcal{A}$ be a finite family of regions that avoid p, with weights $w_A \geq 0$ of total μ. Let W_2 be the sum of $w_A w_{A'}$ over the ordered pairs (A, A') of disjoint regions of $\mathcal{A}$ with $\ell(A, A') = 2$. Let N, e and M be numbers such that the following holds for every $A \in \mathcal{A}$. At most N vertices outside A have a neighbour in A. The regions of $\mathcal{A}$ that are disjoint from A and joined to A by an edge have total weight at most e. For every vertex $u \notin A$ in the component of p in $G - A$, the regions of $\mathcal{A}$ that are disjoint from A, avoid u, and separate u from p in $G - A$ have total weight at most M. Then

$$W_2 \leq 2\mu(e + NM).$$

Proof. Let $A, A' \in \mathcal{A}$ be disjoint. Then A' lies in the component of p in $G - A$, or A lies in the component of p in $G - A'$. Indeed, suppose that the first statement is false. A path from p to its first vertex in A stays in the component of p in $G - A$ before it ends, so it avoids A'. As A is connected and disjoint from A', all of A lies in the component of p in $G - A'$.

Call an ordered pair (A, A') *good* if the regions are disjoint, $\ell(A, A') = 2$, no edge joins them, and A' lies in the component of p in $G - A$. By the last paragraph, the pairs with the first, the second and the fourth property have total weight at least $W_2/2$. The pairs that are joined by an edge have total weight at most $e\mu$. So the good pairs have total weight at least $W_2/2 - e\mu$.

Let (A, A') be good. No edge joins the two regions, so the two links are components Y_1, Y_2 of $G - A - A'$ that are joined to both. The vertex p lies in at most one of them. Let Y be one that avoids p, and let u be a vertex of Y with a neighbour in A. The set Y is connected and joined to A', and A' lies in the component of p in $G - A$. So u lies in that component as well. And Y is a component of $G - A - A'$ that contains u and not p. So A' avoids u and separates u from p in $G - A$. For a given A there are at most N vertices u outside A with a neighbour in A, and for each of them these regions A' have total weight at most M. So the good pairs have total weight at most μNM, and $W_2/2 - e\mu \leq \mu NM$. ■

Proposition 7.10 (Heavy families)**.** Let G be a connected cubic multigraph, and let P be a nonempty set of vertices such that $G[P]$ is connected, $G - P$ has girth at least five, and $q(G,P) \geq \frac{2}{5}$. Let $\mathcal{A}$ be a finite family of regions that avoid P, with weights $w_A \geq 0$ of total μ, and let W_2 be as in Lemma 7.9. Let $\Lambda \geq 4$ be an integer, and let $m \geq 1$, $\kappa > 0$ and $K \geq 0$ be numbers such that

(a) every region of $\mathcal{A}$ has at most Λ boundary edges and at least m vertices;

(b) for every $A \in \mathcal{A}$, the regions of $\mathcal{A}$ that meet A have total weight at most κ;

(c) some random variable $Z \geq 0$ has $\mathbb{E}Z = \mu$ and $\mathbb{E}Z^2 \leq \mu^2 + W_2 + K\mu$, and $Z = 0$ whenever $\xi \cap E(G-P)$ is a forest.

Then

$$\mu \;\leq\; \tfrac{3}{2}K + 3\Lambda\kappa\,\bigl(\Lambda + \nu_\Lambda(G,P) + 3\bigr)\bigl(\lceil 2\Lambda\,\Phi(\Lambda)/m\rceil + 1\bigr).$$

Proof. Put $\nu = \nu_\Lambda(G,P)$ and $w = \lceil 2\Lambda\,\Phi(\Lambda)/m\rceil + 1$. Fix $p \in P$. We apply Lemma 7.9. Let $A \in \mathcal{A}$. By (a), at most Λ vertices outside A have a neighbour in A, and a region that is joined to A by an edge contains one of them. The regions of $\mathcal{A}$ that contain a given vertex all meet one of them, so their total weight is at most κ by (b). Hence we may take $N = \Lambda$ and $e = \Lambda\kappa$. Let $u \notin A$ be a vertex in the component of p in $G - A$, and consider the regions of $\mathcal{A}$ that are disjoint from A, avoid u, and separate u from p in $G - A$. Take a maximal family $C_1, \ldots, C_T$ of pairwise disjoint regions among them. Each of these regions meets some C_i, so their total weight is at most $T\kappa$ by (b). As $q(G,P) > \frac{9}{25}$, Proposition 7.8 gives $T \leq (\Lambda + \nu + 2)\,w$. So we may take $M = (\Lambda + \nu + 2)\,w\kappa$, and the lemma gives $W_2 \leq 2\mu\Lambda\kappa\,(\Lambda + \nu + 3)\,w$.

By (c), $\mathbb{P}(Z \neq 0) \leq 1 - q(G,P) \leq \frac{3}{5}$, and the Cauchy–Schwarz inequality $\mu^2 \leq \mathbb{E}Z^2 \cdot \mathbb{P}(Z \neq 0)$ gives $\mu^2 \leq \frac{3}{5}(\mu^2 + W_2 + K\mu)$. So $\mu \leq \frac{3}{2}(K + W_2/\mu)$ if $\mu > 0$, and this is the claim. ∎

7.4 Packing of short cycles

IDEA | the purpose of the packing bound

The second-moment argument requires a graph of girth about $\log r$. So we delete a maximal family of disjoint short cycles, and we need a bound for the size of this family. Lemma 2.4 gives such a bound only when the proportion of forests exceeds one half by a fixed margin. Here Proposition 7.10 gives a bound for the cycles of length at most d. For large d it contains the number of disjoint cycles of length about $\log d$, and for this number we use induction.

Proposition 7.11 (Packing). Let G be a connected cubic multigraph, and let P be a nonempty set of vertices such that $G[P]$ is connected. Suppose that $q(G,P) \geq \frac{2}{5}$ and that $G - P$ has girth at least five. Let $d \geq 5$ be an integer, and put $C_3 = 8C_2$. Then

(i) $G - P$ has at most $C_3\, 2^d d^{\,C_0+9}$ pairwise disjoint cycles of length at most d;

(ii) $\nu_d(G,P) \leq 2\, C_3\, d^{\,C_0+1}$.

Proof. We use induction on d. Suppose that (i) holds for every smaller integer that is at least five.

The shorter cycles. If $d < \Lambda_0$, no cycle is d-short. Let $d \geq \Lambda_0$, and let g be the largest integer that is less than $C_0 \log(d+2)$. Then $g < d$, and a cycle is d-short if and only if its length is at most g. If $g < 5$, then $G - P$ has no such cycle. Otherwise (i) for g and the choice of Λ_0 give

$$\nu_d(G,P) \;\leq\; C_3\, 2^g g^{\,C_0+9} \;\leq\; C_3\,(d+2)^{C_0}\big(C_0 \log(d+2)\big)^{C_0+9} \;\leq\; 2\, C_3\, d^{\,C_0+1}.$$

This is (ii).

The cycles of length at most d. Let $Q_1, \ldots, Q_n$ be pairwise disjoint cycles of length at most d in $G - P$, and put $X_i = V(Q_i)$. Every vertex of Q_i lies on exactly one edge outside Q_i, and Q_i has at least five vertices. So X_i is a cyclic region with at most d boundary edges and at least five vertices. Let E_i be the event that $\boldsymbol{O}_{X_i}$ holds and $\xi \cap E(G[X_i]) \neq \emptyset$. Put $p_i = \mathbb{P}(E_i)$, let Z be the number of the events E_i that occur, and put $\mu = \mathbb{E}Z$. By Lemma 2.2, $p_i = \mathbb{P}(\boldsymbol{O}_{X_i})\big(1-2^{-\beta(G[X_i])}\big) \geq 2^{-d-1}$ and $\mathbb{P}(E_i \cap E_j) = 2^{\ell(X_i,X_j)-1} p_i p_j$ for $i \neq j$. If $\ell(X_i,X_j) \geq 3$, we use $\mathbb{P}(E_i \cap E_j) \leq p_i$; by Lemma 2.3 this happens for at most d indices j, for each i. Let W_2 be the sum of $p_i p_j$ over the ordered pairs (i,j) with $\ell(X_i,X_j) = 2$. Then $\mathbb{E}Z^2 \leq \mu^2 + W_2 + (d+1)\mu$. If $\xi \cap E(G-P)$ is a forest, then no event E_i occurs, and $Z = 0$.

So Proposition 7.10 applies to the sets X_i with the weights p_i, and with $\Lambda = d$, $m = 5$, $\kappa = 1$ and $K = d+1$; property (b) holds because the sets are pairwise disjoint and $p_i \leq 1$. If $d < \Lambda_0$, then $\nu_d(G,P) = 0$ and $\Phi(d) = C_2$, so

$$\mu \;\leq\; \tfrac{3}{2}(d+1) + 3d\,(d+3)\big(\lceil 2dC_2/5 \rceil + 1\big) \;\leq\; 4\, C_2\, d^3,$$

and $n \leq 2^{d+1}\mu \leq C_3\, 2^d d^3$. If $d \geq \Lambda_0$, then $\lceil 2C_1 d^5/5 \rceil + 1 \leq C_1 d^5$ and $d + 3 \leq d^{\,C_0+1}$. With (ii),

$$\mu \;\leq\; \tfrac{3}{2}(d+1) + 3d\,\big(d + 3 + 2\, C_3\, d^{\,C_0+1}\big)\, C_1 d^5 \;\leq\; 10\, C_1 C_3\, d^{\,C_0+7}.$$

Hence $n \leq 2^{d+1}\mu \leq 20\, C_1 C_3\, 2^d d^{\,C_0+7} \leq C_3\, 2^d d^{\,C_0+9}$, by the choice of Λ_0. This is (i). ∎

7.5 Proof of Theorem 7.1

Let G and P be as in the theorem, and suppose that $q(G,P) \geq \frac{2}{5}$. We derive a contradiction when r is large. All limits below are for $r \to \infty$, and they are uniform in G and P. Put

$$x = \log r, \qquad A = 6\, e^9, \qquad L = \lfloor Ax \rfloor, \qquad D = \lceil x/16 \rceil.$$

Cycles of moderate length. Let S be the union of the vertex sets of a maximal family of pairwise disjoint cycles of length at most D in $G - P$. By Proposition 7.11(i), $|S| \le C_3\, 2^D D^{\,C_0+10} \le r/x^2$ for large r. We use Proposition 2.5 for $\Gamma = G, P, D, L$ and S. Its hypotheses hold for large r: the set P has at most $3x$ vertices and $G[P]$ is connected, $x/16 \le D \le x/8$, and $7x \le L \le x\sqrt{\log x}$. Let $J = G - (P \cup S)$, $\mathcal{C}$, a_C, μ, W_2 and Z be as in the proposition, and put $\eta = r^{-1/60}$. Then

$$\mu \;\ge\; \frac{1}{4}\ln\frac{L}{6x} - \frac{1}{\sqrt{x}} \;>\; 2, \qquad \mathbb{E}Z^2 \;\le\; \mu^2 + W_2 + (1+\eta)\mu.$$

The bound $\mu > 2$ holds because $L/(6x) \ge e^9 - 1$.

The contradiction. We apply Proposition 7.10 to the vertex sets of the cycles of $\mathcal{C}$, with the weights a_C, and with $\Lambda = L$, $m = x/16$, $\kappa = \eta$ and $K = 1 + \eta$. A cycle $C \in \mathcal{C}$ has at most L vertices, each of them has one edge end outside C, and the girth of J is greater than $D \ge x/16$, by part (i) of Proposition 2.5. So $V(C)$ has at most L boundary edges and at least $x/16$ vertices; this is (a). A cycle of $\mathcal{C}$ that meets C passes through one of the at most L vertices of C, so (b) holds by part (ii). If $\xi \cap E(G - P)$ is a forest, then ξ contains no cycle of J, and $Z = 0$ by part (iv); this is (c). For large r we have $\Phi(L) = C_1 L^4$, and $\nu_L(G, P) \le 2C_3 L^{C_0+1}$ by Proposition 7.11(ii). So

$$\mu \;\le\; \tfrac{3}{2}(1+\eta) + 3L\eta\left(L + 2C_3L^{C_0+1} + 3\right)\left(\lceil 32C_1L^5/x\rceil + 1\right) \;=\; \tfrac{3}{2} + o(1),$$

because $L \le Ax$ and $\eta = r^{-1/60}$. This contradicts $\mu > 2$, and Theorem 7.1 is proved. ∎

8 Proof of Theorem 1.1

Let K_0 be as in Section 6.2. By (1.1),

$$H(r) \;\ge\; 2^{r-1} + (r-2)\,2^{r/2-1} \qquad \text{for all } r \ge 3. \tag{8.1}$$

Choose $R \ge 9$ so large that the following holds for every $r \ge R$: Proposition 6.3, Theorem 6.6 and Theorem 7.1 apply to the rank r; and $r - 2 > 10K_0^2$.

Let $r \ge R$. A ring R_r has $H(r)$ cycles by Corollary 6.2, so $\psi(r) \ge H(r)$. By Lemma 3.1, some connected cubic multigraph G of rank r has $\psi(r)$ cycles. We show that G is a ring or that $c(G) \le 2^{r-1} + 5K_0^2\, 2^{r/2}$. In the first case $c(G) \le H(r)$ by Corollary 6.2. In the second case $c(G) < H(r)$ by (8.1), because $r - 2 > 10K_0^2$. So $\psi(r) \le H(r)$.

Case 1: G has a cyclic cut of size at most three. Then $c(G) \le 2^{r-1} + 5K_0^2\, 2^{r/2}$ by Proposition 6.3.

Case 2: G has no cyclic cut of size at most three, and it has a cycle of length four. Then G is a ring or $c(G) < 2^{r-1}$, by Theorem 6.6.

Case 3: G has no cyclic cut of size at most three and no cycle of length four. Let Z be a cycle with at most three edges. Then at most three edges leave $V(Z)$, and at most six edges have an end in $V(Z)$. So $G - V(Z)$ has rank at least $r - 6 > 0$, and $\delta V(Z)$ is a cyclic cut of size at most three. Hence G has girth at least five. The graph G has $2r - 2 \ge 8$ vertices. Choose a set P of eight vertices such that $G[P]$ is connected. Then $|P| \le 3\log r$, and $G - P$ has girth at least five. So $q(G, P) < \frac{2}{5}$ by Theorem 7.1. The graph G has no bridge, because a cut of size one is cyclic (Section 6). So Lemma 3.2 gives $c(G) < \frac{2}{5}\,2^r + 2^{r-5} < 2^{r-1}$. ∎

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